Algebra and calculus review for theoretical ecology
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Algebra

The building blocks for solving for x, showing the intermediate step that applies the same action to both sides of each equation, are:

Problem	$x+a=b$	$ax=b$	$x/a=b$	$x^a=b$
	$(x+a)-a=b-a$	$(ax)/a=b/a$	$(x/a)\cdot a=b\cdot a$	$(x^a)^{1/a}=b^{1/a}$
Solution	$x=b-a$	$x=b/a$	$x=ba$	$x=b^{1/a}$

ln is the natural log, where if $\ln(a)=b$ then $a=e^b$ (and $\ln(e)=1$). Helpful rules for natural log are:

$$\begin{aligned}\ln(ab) &= \ln(a)+\ln(b)\\ \ln(a^b) &= b\ln(a)\\ \ln(1/a)=\ln(a^{-1}) &= -\ln(a).\end{aligned}$$

Therefore, when needing to rearrange equations with ln or e,

Problem	$\ln(x)=b$	$e^x=b$
	$e^{\ln(x)}=e^b$	$\ln(e^x)=\ln(b)$
Solution	$x=e^b$	$x=\ln(b)$

Useful rules for dealing with powers are

$$x^a x^b = x^{a+b} \qquad x^{-1}=\frac{1}{x}$$
$$(x^a)^b = x^{ab} \qquad x^0 = 1.$$

Note that given $e^0=1$, $\ln(1)=0$.
Putting these building blocks together:

- Focus on what you are trying to solve for – in the examples here, x
- If you have fractions, it is typically a good idea to get everything under a common denominator, e.g.,
$$\frac{a}{b}+\frac{c}{d}=\frac{a}{b}\cdot\frac{d}{d}+\frac{c}{d}\cdot\frac{b}{b}=\frac{ad+cb}{bd}$$
such that you can then multiply both sides of your equation by that denominator to get rid of the fractions. Remember that
$$\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c} \text{ but } \frac{a}{b+c}\neq\frac{a}{b}+\frac{a}{c}.$$
- If if what you are solving for has the same power everywhere (e.g., all x's with no x^2's, x^3's, etc.), then first get everything with x on one side and everything else on the other, and second isolate x, using the building blocks above. Here are three examples:

$$\begin{array}{ccc} ax+b=c & ax+b=cx & ax+b=cx+d \\ ax=c-b & b=cx-ax & ax+b-cx=d \\ x=(c-b)/a & b=(c-a)x & ax-cx=d-b \\ & x=b/(c-a) & (a-c)x=d-b \\ & & x=(d-b)/(a-c) \end{array}$$

- When you have multiple powers, look for things to factor. If you have an expression that $= 0$ and something is factorable that you can assume is nonzero (e.g., a parameter, such as population growth rate, that has to be positive), then divide it out. If something is factorable that you cannot assume is nonzero (e.g., a state variable, such as population size, that can be zero or positive), then split the factors and set each to zero separately as two possible solutions: if $f(x)g(x) = 0$, then both $f(x) = 0$ and $g(x) = 0$ are possible solutions. For example:

$$\begin{aligned} ax^2 - x &= 0 \\ x(ax-1) &= 0 \\ x = 0 \quad \text{or} \quad ax - 1 &= 0 \\ x &= 1/a \end{aligned}$$

  In particular, if you have an expression $xf(x) = 0$ (often the case when solving a continuous-time model for the equilibrium) or $xf(x) = x$ (often the case when solving a discrete-time model for the equilibrium), when you divide by x to simplify, $x = 0$ is a possible alternative solution. Be sure to note this unless a question specifies that you are looking for the nonzero solution/equilibrium. Conversely, $x = 0$ is only a possible equilibrium if the entire expression is multiplied by x such that you can factor it out.

- If you have a quadratic equation, $ax^2 + bx + c = 0$, it has two solutions,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  although it is always a good idea to check if you can factor your equation easily first before applying this.

- When you have two equations and two unknowns, first rearrange one equation so one unknown is in terms of the other. Then plug that expression into the other equation, solve that equation for the one remaining unknown, and go back to your original equation to solve for the other unknown. For example, solving for x and y:

$$\begin{array}{rcl} 2x - y = 0 & \text{and} & x + y = 1 \\ y = 2x & \rightarrow & x + 2x = 1 \\ y = 2(1/3) & & 3x = 1 \\ y = 2/3 & \nwarrow & x = 1/3 \end{array}$$

Calculus

Definition of a derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x},$$

the rate of change or slope of f at x.

Properties of derivatives: $f'(x) = \frac{df(x)}{dx}$

Constant factor	$\frac{d}{dx} cf(x) = cf'(x)$
Addition	$\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$
Multiplication (product rule)	$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
Division (quotient rule)	$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
Subfunction (chain rule)	$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

Common derivatives

$$\frac{d}{dx} c = 0 \qquad \frac{d}{dx} \ln(x) = \frac{1}{x}$$

$$\frac{d}{dx} cx = c \qquad \frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx} x^n = nx^{n-1} \qquad \frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)}$$

Properties of integrals

Constant factor	$\int cf(x)dx = c\int f(x)dx$
Addition	$\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
Integration by parts	$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$

Common integrals

$$\int c dx = cx \qquad \int \frac{1}{x} dx = \ln(x)$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} \qquad \int e^{cx} dx = \frac{1}{c} e^{cx}$$

Taylor expansion

$$f(x + c) = f(c) + xf'(c) + \frac{x^2}{2!} f''(c) + \frac{x^3}{3!} f'''(c) + ... + \frac{x^n}{n!} f^{(n)}(c) + ...$$

Linear algebra

Note: We will cover this material in class, as a linear algebra course is not a prerequisite.

A **vector** is a row or column of numbers, for example

$$\vec{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \text{ or } \vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

A **matrix** is a rectangular array of numbers, for example

$$M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

A **scalar** is a term not in any array, for example c.

Matrix addition and subtraction is element-by-element; matrices (or vectors) need to be the same size to be added or subtracted to each other. For example:

$$\begin{aligned} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} &= \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \end{bmatrix} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} &= \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}. \end{aligned}$$

When a scalar is multiplied by a matrix or vector, it is multiplied by each element, for example

$$\begin{aligned} c \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} cv_1 \\ cv_2 \end{bmatrix} \\ c \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}. \end{aligned}$$

Matrix multiplication happens across rows and down columns: for matrix A with elements a_{ij} and B with elements b_{ij}, the ij^{th} element of their product AB is $\sum_k a_{ik} b_{kj}$. Therefore, the inner dimensions of two multiplied matrices much agree (i.e., if A is a $m \times n$ matrix, then B must be a $n \times p$ matrix, and AB has dimensions $m \times p$), and order matters ($AB \neq BA$). For example:

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} &= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}. \end{aligned}$$

The **identity matrix**, I, has ones on the diagonal and zeros elsewhere, for example

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The identity matrix multiplied by any vector or matrix is that same vector or matrix (i.e., $I\vec{v} = \vec{v}$ and $IM = M$).

The **trace** of a matrix is the sum of the elements on its diagonal, for example

$$Tr \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11} + a_{22}.$$

The **determinant** of a matrix in the 2×2 case is

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

An **eigenvalue** (λ) and **eigenvector** ($\vec{v}$) of a matrix are a scalar and vector such that

$$M\vec{v} = \lambda\vec{v}.$$

To solve for the eigenvalues and eigenvectors of a matrix, first rearrange

$$\begin{aligned} M\vec{v} - \lambda\vec{v} &= 0 \\ M\vec{v} - \lambda I\vec{v} &= 0 \\ (M - \lambda I)\vec{v} &= 0. \end{aligned}$$

This is true only if the determinant

$$|M - \lambda I| = 0.$$

In the 2×2 case, this is

$$\begin{aligned} \left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| &= 0 \\ \left| \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| &= 0 \\ \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} &= 0 \\ (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} &= 0 \\ \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) &= 0 \\ \lambda &= \frac{a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}. \end{aligned}$$

This yields two possible values for the eigenvalue λ (an $n \times n$ matrix will have n eigenvalues). To find the corresponding eigenvector for each, return to the original definition of eigenvalues and eigenvectors

$$\begin{aligned} M\vec{v} &= \lambda\vec{v} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\ \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix} &= \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \end{bmatrix}, \end{aligned}$$

or

$$\begin{aligned} a_{11}v_1 + a_{12}v_2 &= \lambda v_1 \\ a_{21}v_1 + a_{22}v_2 &= \lambda v_2. \end{aligned}$$

Because of the definition of eigenvalues and eigenvectors, if any vector $\vec{v}$ solves $M\vec{v} = \lambda\vec{v}$, then a constant c times that vector $\vec{u} = c\vec{v}$ is also an eigenvector ($Mc\vec{v} = \lambda c\vec{v}$ so $M\vec{u} = \lambda\vec{u}$). Therefore, you will find the solution to be an expression of v_1 relative to v_2, i.e., $v_2 = av_1$, such that your eigenvector is $\begin{bmatrix} v_1 \\ av_1 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ a \end{bmatrix}$. One approach is to set one entry to one such that the other entry is expressed relative to it (e.g., let $v_1 = 1$). Another approach is to set the sum to one ($v_1 + v_2 = 1$) to normalize (e.g., when expressing a stable age distribution in terms of the proportion in each age class).

Equilibrium terminology

- "Equilibrium" means no change in time. In continuous time ($\frac{dn}{dt} = f(n)$), this means the change over time is zero ($f(\bar{n}) = 0$). In discrete time ($N_{t+1} = F(N_t)$), this means that the population size in the next time step is equivalent to that in the previous ($\bar{N} = F(\bar{N})$).
- "Biologically relevant" equilibrium means that the equilibrium population size non-negative (zero or positive) and real (not imaginary).
- In a two-species case, the "zero" equilibrium has both species equal to zero, the "edge" equilibrium means one species is equal to zero and the other is nonzero (on the edge of a phase-plane plot of one species vs. the other), and the "internal" equilibrium means both are nonzero (in the interior of a phase-plane plot of one species vs. the other).
- An "isocline" is a line along which one species is not changing (e.g., in a two-species case with n_1 and n_2, $\frac{dn_1}{dt} = 0$ without any constraint on $\frac{dn_2}{dt}$).