The Effects of Nature on Learning in Games

C.-Y. Cynthia Lin Lawell[1]

Abstract

This paper develops an agent-based model to investigate the effects of Nature on learning in games. In particular, I extend one commonly used learning model — stochastic fictitious play — by adding a player to the game — Nature — whose play is random and non-strategic, and whose underlying exogenous stochastic process may be unknown to the other players. Because the process governing Nature's play is unknown, players must learn this process, just as they must learn the distribution governing their opponent's play. Nature represents any random state variable that may affect players' payoffs, including global climate change, weather shocks, fluctuations in environmental conditions, natural phenomena, changes to government policy, technological advancement, and macroeconomic conditions. Results show that when Nature is Markov and the game being played is a static Cournot duopoly game, neither play nor payoffs converge, but they each eventually enter respective ergodic sets.

***JEL* Classification:** C72, C73, C87, L13

Keywords: stochastic fictitious play, learning in games, common shock, Nature, agent-based model

This draft: March 2016

---

[1] University of California at Davis; cclin@primal.ucdavis.edu. I am grateful to Aditya Sunderam for excellent research assistance. This paper also benefited from discussions with Gary Chamberlain, Drew Fudenberg, Ariel Pakes, Burkhard Schipper, Karl Schlag, and Sam Thompson, and from comments from participants at the workshop in microeconomic theory at Harvard University and at the Second World Congress of the Game Theory Society. I am a member of the Giannini Foundation of Agricultural Economics. All errors are my own.

1. Introduction

Although most work in noncooperative game theory has traditionally focused on equilibrium concepts such as Nash equilibrium and their refinements such as perfection, models of learning in games are important for several reasons. The first reason why learning models are important is that mere introspection is an insufficient explanation for when and why one might expect the observed play in a game to correspond to an equilibrium. For example, experimental studies show that human subjects often do not play equilibrium strategies the first time they play a game, nor does their play necessarily converge to the Nash equilibrium even after repeatedly playing the same game (see e.g., Erev and Roth, 1998).

In contrast to traditional models of equilibrium, learning models appear to be more consistent with experimental evidence (Fudenberg and Levine, 1999). These models, which explain equilibrium as the long-run outcome of a process in which less than fully rational players grope for optimality over time, are thus potentially more accurate depictions of actual real-world strategic behavior. For example, in their analysis of a newly created market for frequency response within the UK electricity system, Doraszelski, Lewis and Pakes (2016) show that models of fictitious play and adaptive learning predict behavior better than Nash equilibrium prior to convergence. By incorporating exogenous common shocks, this paper brings these learning theories even closer to reality.

In addition to better explaining actual strategic behavior, the second reason why learning models are important is that they can be useful for simplifying computations in empirical work. Even if they are played, equilibria can be difficult to derive analytically and computationally in real-world games. For cases in which the learning dynamics converge to an equilibrium, deriving the equilibrium from the learning model may be computationally less burdensome than attempting to solve for the equilibrium directly. Indeed, the fictitious play learning model was first introduced as a method of computing Nash equilibria (Hofbauer and Sandholm, 2001). Pakes and McGuire (2001) use a model of reinforcement learning to reduce the computational burden of calculating a single-agent value function in their algorithm for computing symmetric Markov perfect equilibria. Lee and Pakes (2009) examine how different learning algorithms "select out" equilibria when multiple equilibria are possible. As will be explained below, the work presented in this paper further enhances the applicability of these learning models to empirical work.

In this paper, I extend one commonly used learning model—stochastic fictitious play—by adding a player to the game—Nature—whose play is random and non-strategic, and whose underlying exogenous stochastic process may be unknown to the other players. Because the process governing Nature's play is unknown, players must learn this process, just as they must learn the distribution governing their opponent's play. Nature represents any random state variable that may affect players' payoffs, including global climate change, weather shocks, fluctuations in environmental conditions, natural phenomena, changes to government policy, technological advancement, and macroeconomic conditions.

By adding Nature to a model of stochastic fictitious play, my work makes several contributions. First, incorporating Nature brings learning models one step closer to realism. Stochastic fictitious play was introduced by Fudenberg and Kreps (1993), who extended the standard deterministic fictitious model by allowing each player's payoffs to be perturbed each period by i.i.d. random shocks (Hofbauer and Sandholm, 2001). Aside from these idiosyncratic shocks, however, the game remains the same each period. Indeed, to date, much of the learning literature has focused on repetitions of the same game (Fudenberg and Levine, 1999).[2] My model extends the stochastic fictitious play model one step further by adding a common shock that may not be i.i.d. over time, and whose distribution and existence may not necessarily be common knowledge. Because aggregate payoffs are stochastic, the game itself is stochastic; players do not necessarily play the same game each period. Thus, as in most real-world situations, payoffs are a function not only of the players' strategies and of individual-specific idiosyncratic shocks, but also of common exogenous factors as well. If the players were firms, for example, as will be the case in the particular model I evaluate, these exogenous factors may represent global climate change, weather shocks, fluctuations in environmental conditions, natural phenomena, changes to government policy, technological advancement, macroeconomic fluctuations, or other conditions that affect the profitability of all firms in the market.

The second main contribution my work makes is that it provides a framework for empirical structural estimation. Although the use of structural models to estimate equilibrium dynamics has been popular in the field of empirical industrial organization, little work has been done either estimating structural models of non-equilibrium dynamics or using learning models to reduce the computational burden of estimating equilibrium models. One primary reason learning models have not been ignored by empirical economists thus far is that none of the models to date have allowed for the possibility of stochastic state variables and are thus too unrealistic for estimation.[3] My work removes this obstacle to structural estimation.

My research objective is to investigate the effects of adding Nature to the Cournot duopoly on the stochastic fictitious play dynamics. In particular, I analyze the effects of Nature on:

---

[2] Exceptions are Li Calzi (1995) and Romaldo (1995), who develop models of learning from similar games.
[3] Ariel Pakes, personal communication.

(i) Trajectories*:* What do the trajectories for strategies, assessments, and payoffs look like?

(ii) Convergence: Do the trajectories converge? Do they converge to the Nash equilibrium? How long does convergence take?

(iii) Welfare: How do payoffs from stochastic fictitious play compare with those from the Nash equilibrium? When do players do better? Worse?

(iv) Priors: How do the answers to (i)-(iii) vary when the priors are varied?

Results show that when Nature is first-order Markov and the game being played is a static Cournot duopoly game, the trajectories of play and payoffs are discontinuous and can be viewed as an assemblage of the dynamics that arise when Nature evolves as several separate i.i.d. processes, one for each of the possible values of Nature's previous period play. Neither play nor payoffs converge, but they each eventually enter respective ergodic sets.

The results of this paper have important implications for environmental and natural resource issues such as global climate change for which there is uncertainty about the distribution of possible outcomes, and about which agents such as individuals, firms, and/or policy-makers seek to learn.

The balance of this paper proceeds as follows. I describe my model in Section 2. I outline my methods and describe my agent-based model in Section 3. In Section 4, I analyze the Cournot duopoly dynamics in the benchmark case without Nature. I analyze the results when Nature is added in Section 5. Section 6 concludes.

2. Model

2.1 Cournot duopoly

The game analyzed in this paper is a static homogeneous-good Cournot duopoly.[4] I choose a Cournot model because it is one of the widely used concepts in applied industrial organization (Huck, Normann and Oeschssler, 1999);[5] I focus on two firms only so that the phase diagrams for the best response dynamics can be displayed graphically.[6]

In a one-shot Cournot game, each player i chooses a quantity q_i to produce in order to maximize her one-period profit (or payoff)[7]

$$\pi_i(q_i, q_j) = D^{-1}(q_i + q_j)q_i - C_i(q_i)$$

where $D^{-1}(\bullet)$ is the inverse market demand function and $C_i(q_i)$ is the cost to firm i of producing q_i. Each firm i's profit-maximization problem yields the best-response function:

$$BR_i(q_j) = \arg\max_{q_i} \pi_i(q_i, q_j)$$

I assume that the market demand $D(\bullet)$ for the homogeneous good as a function of price p is linear and is given by:

$$D(p) = a - bp$$

where $a \geq 0$ and $b \geq 0$. I assume that the cost $C_i(\bullet)$ to each firm i of producing q_i is quadratic and is given by:

$$C_i(q_i) = c_i q_i^2$$

[4] Although the particular game I analyze in this paper is a Cournot duopoly, my agent-based model can be used to analyze any static normal-form two-player game.
[5] For experimental studies of learning in Cournot oligopoly, see Huck, Normann and Oeschssler (1999).
[6] Although my agent-based model can only generate phase diagrams for two-player games, it can be easily modified to generate other graphics for games with more than two players.
[7] Throughout this paper, I will use the terms "profit" and "payoff" interchangeably. Both denote simply the one-period profit.

where $c_i \geq 0$. With these assumptions, the one-period payoff to each player i is given by:

$$\pi_i(q_i, q_j) = \left(\frac{a}{b} - \frac{q_i + q_j}{b}\right) q_i - c_i q_i^2 \quad ,$$

the best-response function for each player i is given by:

$$BR_i(q_j) = \frac{(a - q_j)}{2(1 + c_i b)}$$

and the Nash equilibrium quantity for each player i is given by:

$$q_i = \frac{a(1 + 2c_j b)}{4(1 + c_i b)(1 + c_j b) - 1} \quad .$$

Throughout the simulations, I set a = 20, b = 1. With these parameters, the maximum total production $\bar{q}$, corresponding to p = 0, is $\bar{q} = 20$. The pure-strategy space S^i for each player i is thus the set of integer quantities from 0 to $\bar{q} = 20$. I examine two cases in terms of cost functions. In the symmetric case, I set $c_1 = c_2 = 1/2$; in the asymmetric case, the higher-cost player 1 has $c_1 = 4/3$, while the lower-cost player 2 has $c_2 = 5/12$. The Nash equilibrium quantities are thus q_1^{NE} = 5, q_2^{NE} = 5 in the symmetric case and q_1^{NE} = 3, q_2^{NE} = 6 in the asymmetric case. These correspond to payoffs of π_1^{NE} = 37.5, π_2^{NE} = 37.5 in the symmetric case and π_1^{NE} = 21, π_2^{NE} = 51 in the asymmetric case. The monopoly profit or, equivalently, the maximum joint profit that could be achieved if the firms cooperated, is π^m = 80 in the symmetric case and π^m =75.67 in the asymmetric case.[8]

[8] As a robustness check, I also run all the simulations under an alternative set of cost parameters. The alternative set of parameters in the symmetric cost case are $c_1 = c_2 = 0$, which yields a Nash equilibrium quantity of $q_1^{NE} = q_2^{NE}$ = 5 and a Nash equilibrium payoff of $\pi_1^{NE} = \pi_2^{NE}$ = 37.5. The alternative set of parameters in the asymmetric cost case are $c_1 = 0.5$, $c_2 = 0$, which yields Nash equilibrium quantities of q_1^{NE} = 4, q_2^{NE} = 8 and Nash equilibrium payoffs of

2.2 Logistic smooth fictitious play

The one-shot Cournot game described above is played repeatedly and the players attempt to learn about their opponents over time. The learning model I implement is that of stochastic fictitious play. In fictitious play, agents behave as if they are facing a stationary but unknown distribution of their opponents' strategies; in stochastic fictitious play, players randomize when they are nearly indifferent between several choices (Fudenberg and Levine, 1999). The particular stochastic play procedure I implement is that of logistic smooth fictitious play.

Although the one-shot Cournot game is played repeatedly, I assume, as is standard in learning models, that current play does not influence future play, and therefore ignore collusion and other repeated game considerations. As a consequence, the players regard each period-*t* game as an independent one-shot Cournot game. There are several possible stories for why it might be reasonable to abstract from repeated play considerations in this duopoly setting. One oft-used justification is that each period there is an anonymous random matching of the firms from a large population of firms (Fudenberg and Levine, 1999). This matching process might represent, for example, random entry and/or exit behavior of firms. It might also depict a series of one-time markets, such as auctions, the participants of which differ randomly market by market. A second possible story is that legal and regulatory factors may preclude collusion.

For my model of logistic smooth fictitious play, I use notation similar to that used in Fudenberg and Levine (1999). As explained above, the pure-strategy space S^i for each player i is

---

$\pi_1^{NE} = 24$, $\pi_2^{NE} = 64$. Except where noted, the results across the two sets of parameters have similar qualitative features.

the set of integer quantities from 0 to $\bar{q} = 20$. A pure strategy q_i is thus an element of this set: $q_i \in S^i$. The per-period payoff to each player *i* is simply the profit function $\pi_i(q_i, q_j)$.[9]

At each period *t*, each player *i* has an assessment $\gamma_t^i(q_j)$ of the probability that his opponent will play q_j. This assessment is given by

$$\gamma_t^i(\tilde{q}_j) = \frac{\kappa_t^i(\tilde{q}_j)}{\sum_{q_j=0}^{\bar{q}} \kappa_t^i(q_j)},$$

where the weight function $\kappa_t^i(q_j)$ is given by:

$$\kappa_t^i(q_j) = \kappa_{t-1}^i(q_j) + \mathrm{I}\{q_{j,t-1} = q_j\}$$

with exogenous initial weight function $\kappa_0^i(q_j): S^j \rightarrow \Re_+$. Thus, for all periods *t*, γ_t^i, κ_t^i and κ_0^i are all $1 \times \bar{q}$ vectors. For my various simulations, I hold the length of the fictitious history, $\sum_{q_j=0}^{\bar{q}} \kappa_t^i(q_j)$, constant at $\bar{q} + 1 = 21$ and vary the distribution of the initial weights.

In logistic smooth fictitious play, at each period *t*, given her assessment $\gamma_t^i(q_j)$ of her opponent's play, each player *i* chooses a mixed strategy σ_i so as to maximize her perturbed utility function:

$$\tilde{U}^i(\sigma_i, \gamma_t{}^i) = E_{q_i, q_j}[\pi_i(q_i, q_j) \mid \sigma_i, \gamma_t{}^i] + \lambda v^i(\sigma_i),$$

where $v^i(\sigma_i)$ is an admissible perturbation of the following form:

$$v^i(\sigma_i) = \sum_{q_i=0}^{\bar{q}} -\sigma_i(q_i) \ln \sigma_i(q_i).$$

[9] In Fudenberg and Levine (1999), a pure strategy is denoted by s^i instead of q_i, and payoffs are denoted by $u^i(s^i, s^j)$ instead of $\pi_i(q_i, q_j)$.

With these functional form assumptions, the best-response distribution $\overline{BR^i}$ is given by

$$\overline{BR^i}(\gamma_t^i)[\tilde{q}_i] = \frac{\exp(1/\lambda)\,\mathrm{E}_{q_i}[\pi_i(\tilde{q}_i, q_j) \mid \gamma_t^i]}{\sum_{q_i=0}^{\bar{q}} \exp(1/\lambda)\,\mathrm{E}_{q_i}[\pi_i(q_i, q_j) \mid \gamma_t^i]} .$$

The mixed strategy θ_t^i played by player *i* at time *t* is therefore given by:

$$\theta_t^i = \overline{BR^i}(\gamma_t^i) \quad .$$

The pure action q_{it} actually played by player *i* at time *t* is drawn for player *i*'s mixed strategy:

$$q_{it} \sim \theta_t^i .$$

Because each of the stories of learning in static duopoly I outlined above suggest that each firm only observes the play of its opponent and not the plays of other firms of the opponent's "type" in identical and simultaneous markets, I assume that each firm only observes the actual pure-strategy action q_{it} played by its one opponent and not the mixed strategy θ_t^i from which that play was drawn.

I choose the logistic model of stochastic fictitious play because of its computational simplicity and because it corresponds to the logit decision model widely used in empirical work (Fudenberg and Levine, 1999). For the simulations, I set $\lambda = 1$.

2.3 Adding Nature

I now extend the stochastic fictitious play model by adding a player to the game—Nature—whose play is random and non-strategic, and whose underlying exogenous stochastic process may be unknown to the other players. Because the process governing Nature's play is unknown, players must learn this process, just as they must learn the distribution governing their opponent's play. Nature represents any random state variable that may affect players' payoffs, including global climate change, weather shocks, fluctuations in environmental conditions, natural phenomena, changes to government policy, technological advancement, and macroeconomic conditions.

In particular, I model Nature's "play" at time t as a shock ε_t to the demand function such that the actual demand function at time t is:

$$D(p) = a - bp + \varepsilon_t .$$

As a result, each player i's one-period payoff is given by:

$$\pi_i(q_i, q_j, \varepsilon_t) = \left(\frac{a + \varepsilon_t}{b} - \frac{q_i + q_j}{b} \right) q_i - c_i q_i^2 \quad .$$

In the particular model of Nature I consider, Nature behaves as a Markov process with support $N \equiv \{-4, 0, +4\}$ and with a Markov transition matrix M given by:

$$\mathrm{M}_{ij} = \Pr(\varepsilon_t = j \mid \varepsilon_{t-1} = i) \quad .$$

For example, if Nature represents global climate change, the shocks may represent weather shocks that may vary year to year following a first-order Markov process.

For my simulations, I use the following specification for M:

		ε_t		
		-4	0	4
	-4	0.80	0.10	0.10
ε_{t-1}	0	0.45	0.10	0.45
	4	0.10	0.10	0.80

where $\varepsilon_{-1} \equiv 0$. This particular Markov transition matrix creates processes that have a high degree of persistence, with long streaks of negative shocks and long streaks of positive shocks. Shocks of value zero are rare and transitory: anytime the shock takes on value zero, it is likely to become either positive or negative.[10]

In the particular model of players' belief about Nature that I consider, players are aware that Nature is Markov and they know the support of Nature's distribution, but they do not know the value of Nature's transition matrix. Each player begins with the prior belief that each Markov transition is equally likely. To learn the distribution of Nature's play, firms update their assessments for each transition probability is a manner analogous to that in fictitious play.

Because the unconditional distribution of Nature has mean zero, I can compare the quantities and payoffs arising from stochastic fictitious play in the presence of Nature with the quantities and payoffs that would arise in Nash equilibrium in the absence of Nature.

3. Methods

To analyze the dynamics of logistic smooth fictitious play, I develop an agent-based model that enables one to analyze the following.

---

[10] I also have preliminary results for two other models of Nature as well (not shown). In the simplest model, Nature's shock in each period is independent and identically distributed (IID) with mean zero and discrete support. In particular, ε_t is drawn randomly from the set $N \equiv \{-4, 0, +4\}$, where each element of N has an equal probability of being drawn.

In the other model of Nature, Nature behaves as an autoregressive process of order 1 (AR(1)). Thus, I define

$$\varepsilon_t = \rho\varepsilon_{t-1} + u_t$$

where $0 < \rho < 1$ and u_t is normally distributed with mean 0 and standard deviation σ^2. ε_0 is drawn from the unconditional marginal distribution of ε_t: a normal distribution with mean 0 and standard deviation $\sigma^2/(1-\rho^2)$. This case is similar to the Markov model because again ε_t is dependent on ε_{t-1}. However, the autoregressive model differs from the previous two since here Nature's shocks take on continuous, rather than discrete, values.

I hope to fully analyze these two additional models of Nature in future work.

Moreover, in future work, in addition to allowing Nature to evolve in three different ways (IID, Markov and AR(1)), one can model players' beliefs about Nature in five different ways. Players can believe that Nature behaves as an IID, Markov, or autoregressive process, or they can be ignorant of the presence of Nature altogether. I also consider the case in which players have asymmetric beliefs about Nature.

(i) Trajectories

For each player *i*, I examine the trajectories over time for the mixed strategies θ_t^i chosen, the actual pure actions q_{it} played and payoffs π_{it} achieved. I also examine, for each player *i*, the trajectories for the per-period mean quantity of each player's mixed strategy:

$$\mathrm{E}[q_i \mid \theta_t^i] \tag{1}$$

as well as the trajectories for the per-period mean quantity of his opponent's assessment of his strategy:

$$\mathrm{E}[q_i \mid \gamma_t^j]. \tag{2}$$

I also examine three measures of the players' payoffs.[11] First, I examine the *ex ante payoffs*, which I define to be the payoffs a player expects to achieve before her pure-strategy action has been drawn from her mixed strategy distribution:

$$\mathrm{E}_{q_i, q_j}[\pi_i(q_i, q_j) \mid \theta_t^i, \gamma_t^i]. \tag{3}$$

The second form of payoffs are the *interim payoffs*, which I define to be the payoffs a player expects to achieve after she knows which particular pure-strategy action q_{it} has been drawn from her mixed strategy distribution, but before her opponent has played:

$$\mathrm{E}_{q_j}[\pi_i(q_{it}, q_j) \mid \gamma_t^i] \tag{4}$$

The third measure of payoffs I analyze is the actual realized payoff $\pi_i(q_{it}, q_{jt})$.

[11] I examine the payoffs (or, equivalently, profits) instead of the perturbed utility so that I can compare the payoff from stochastic fictitious play with the payoffs from equilibrium play.

(ii) Convergence

The metric I use to examine convergence is the Euclidean norm $d(\bullet)$. Using the notion of a Cauchy sequence and the result that in finite-dimensional Euclidean space, every Cauchy sequence converges (Rudin, 1976), I say that a vector-valued trajectory $\{X_t\}$ has converged at time τ if for all $m,n \geq \tau$ the Euclidean distance between its value at periods *m* and *n*, $d(X_m, X_n)$, falls below some threshold value $\bar{d}$. In practice, I set $\bar{d} = 0.01$ and require that $d(X_m, X_n) < \bar{d}$ $\forall m,n \in [\tau, T]$, where T=1000. I examine the convergence of two trajectories: the mixed strategies $\{\theta_t^i\}$ and ex ante payoffs $\{\mathrm{E}_{q_i,q_j}[\pi_i(q_i,q_j) \,|\, \theta_t^i, \gamma_t^i]\}$.

In addition to analyzing whether or not either the mixed strategies or the ex ante payoffs converge, I also examine whether or not they converge to the Nash equilibrium strategy and payoffs, respectively. I say that a vector-valued trajectory $\{X_t\}$ has converged to the Nash equilibrium at time τ if the Euclidean distance between its value at and that of the Nash equilibrium analog, $d(X_t, X^{NE})$, falls below some threshold value $\bar{d}$ for all periods after τ. In practice, I set $\bar{d} = .01$ and require that $d(X_t, X^{NE}) < \bar{d}$ $\forall t \in [\tau, T]$, where T=1000.

(iii) Welfare

The results above are compared to non-cooperative Nash equilibrium as well as the cooperative outcome that would arise if the firms acted to maximize joint profits. The cooperative outcome corresponds to the monopoly outcome.

(iv) Priors

Finally, I examine the effect of varying both the mean and spread of players' priors κ_0, the above results. These priors reflect the initial beliefs each player has about his opponent prior to the start of play.

The agent-based model I develop for analyzing the dynamics of logistic smooth fictitious play can be used for several important purposes. First, this agent-based model enables one to confirm and visualize existing analytic results. For example, for classes of games for which convergence results have already been proven, my agent-based model enables one not only to confirm the convergence, but also to visualize the transitional dynamics. I demonstrate such a use of the agent-based model in my analysis of the benchmark case of stochastic fictitious play in the absence of Nature.

A second way in which my agent-based model can be used is to generate ideas for future analytic proofs. Patterns gleaned from computer simulations can suggest results that might then be proven analytically. For example, one candidate for an analytic proof is the result that, when costs are asymmetric and priors are uniformly weighted, the higher-cost player does better under stochastic fictitious play than she would under the Nash equilibrium. Another candidate is the result is what I term the *overconfidence premium*: the worse off a player initially expects her opponent to be, the better off she herself will eventually be.

A third way in which of my agent-based model can be used is to analyze games for which analytic solutions are difficult to derive. For example, an analysis of the effects of adding Nature is more easily done numerically rather than analytically.

A fourth potential use for my agent-based model is pedagogical. The agent-based model can supplement standard texts and papers as a learning or teaching tool in any course covering learning dynamics and stochastic fictitious play.

I apply the agent-based model to analyze the stochastic fictitious play dynamics of the Cournot duopoly game both in the absence of Nature and in the presence of Markov Nature.[12] Although the entire agent-based model was run for two sets of parameters, I present the results from only one. Unless otherwise indicated, qualitative results are robust across the two sets of parameters.

4. Benchmark case: No Nature

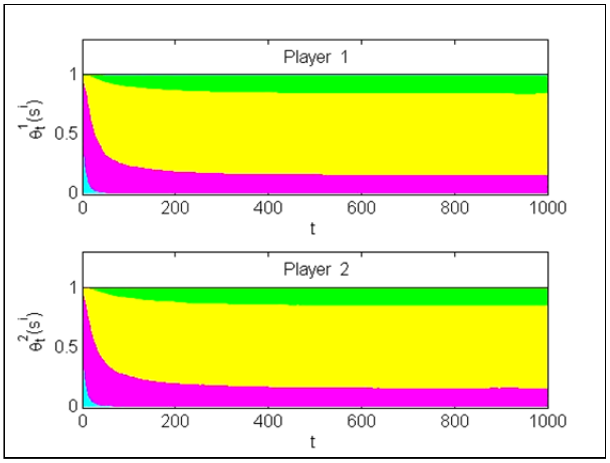
Before adding Nature, I first analyze the stochastic fictitious play dynamics of the Cournot duopoly game in the absence of Nature. I do so for two reasons. First, results from the no Nature case provide a benchmark against which I can compare my results with Nature. Second, since my Cournot duopoly game with linear demand falls into a class of games for which theorems about convergence have already been proven,[13] a presentation of my results enables one not only to confirm the previous proven analytic results, but also to assess how my numerical results may provide additional information and intuition previously inaccessible to analytic analysis alone.

[12] In future work I hope to examine several different scenarios, each corresponding to a different specification of Nature (e.g., no Nature, IID Nature, Markov Nature, AR(1) Nature) and to a different specification of players' beliefs about Nature (e.g., no Nature, IID Nature, Markov Nature, AR(1) Nature, asymmetric beliefs).

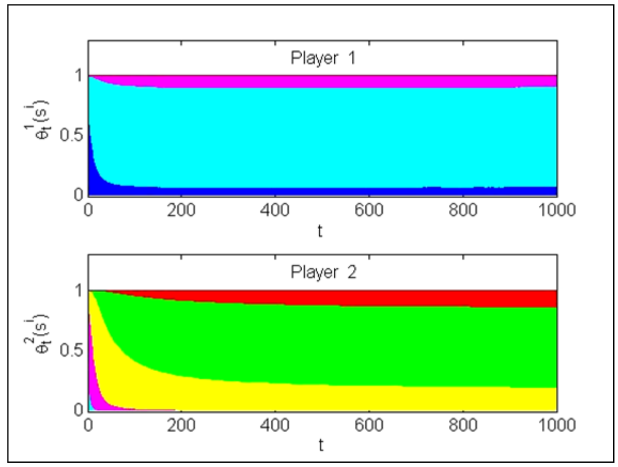
[13] More specifically, because my game is a 2X2 game that has a unique strict Nash equilibrium, the unique intersection of the smoothed best response functions is a global attractor (Fudenberg and Levine, 1999). Leoni (2008) extends the convergence result of Kalai and Lehrer (1993a, 1993b) to a class of games in which players have a payoff function continuous for the product topology, and construct a Nash equilibrium such that, under certain conditions and after some finite time, the equilibrium outcome of learning in repeated games is arbitrarily close to the constructed Nash equilibrium. Arieli and Young (2016) provide explicit bounds on the speed of convergence for the general case of weakly acyclic games with global interaction.

First, I present results that arise when each player initially believes that the other plays each possible pure strategy with equal probability. In this case, each player's prior puts uniform weight on all the possible pure strategies: κ_0^i=(1,1, …, 1) $\forall i$. I call this form of prior a *uniformly weighted prior*. When a player has a uniformly weighted prior, he will expect his opponent to produce quantity 10 on average, which is higher than the symmetric Nash equilibrium quantity of $q_1^{NE} = q_2^{NE}$ = 5 in the symmetric cost case and also higher than both quantities q_1^{NE} = 3, q_2^{NE} = 6 that arise in the Nash equilibrium of the asymmetric cost case.

Figure 1.1 presents the trajectories of each player *i*'s mixed strategy θ_t^i over time when each player has a uniformly weighted prior. Each color in the figure represents a pure strategy (quantity) and the height of the band represents the probability of playing that strategy. As expected, in the symmetric case, the players end up playing identical mixed strategies. In the asymmetric case, player 1, whose costs are higher, produces smaller quantities than player 2. In both cases the players converge to a fixed mixed strategy, with most of the change occurring in the first 100 time steps. It seems that convergence takes longer in the case of asymmetric costs than in the case of symmetric costs. Note that the strategies that eventually dominate each player's mixed strategy initially have very low probabilities. The explanation for this is that with uniformly weighted priors, each player is grossly overestimating how much the other will produce. Each player expects the other to produce quantities between 0 and 20 with equal probabilities, and thus has a mean prior of quantity 10. As a consequence, each firm initially produces much less the Nash equilibrium quantity to avoid flooding the market. In subsequent periods, the players will update their assessments with these lower quantities and change their strategies accordingly.



(a)



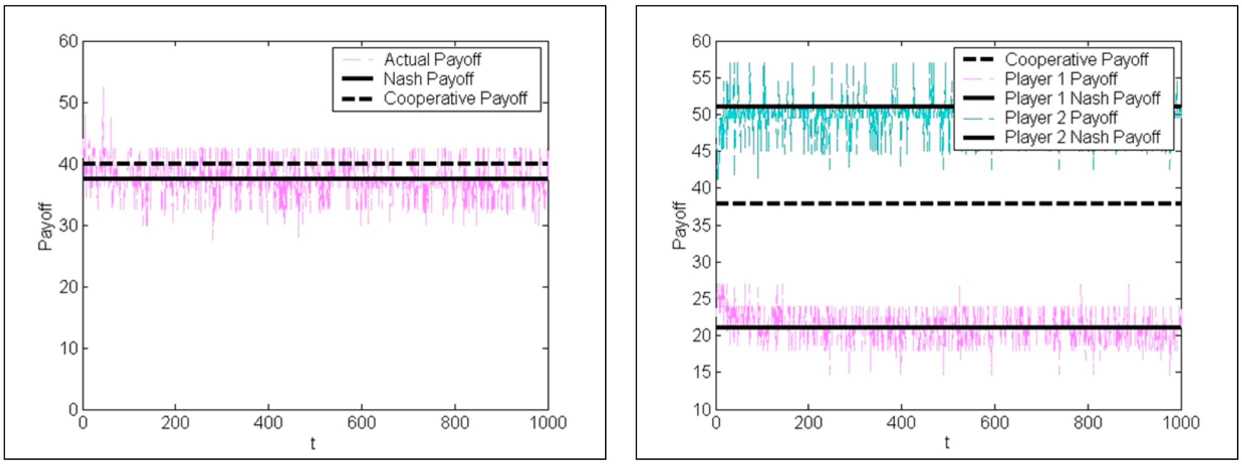
(b)

Figure 1.1. Dynamics of players' mixed strategies with (a) symmetric and (b) asymmetric costs as a function of time in the absence of Nature. As a benchmark, the Nash equilibrium quantities are $q^{NE} = (5,5)$ in the symmetric cost case and $q^{NE} = (3,6)$ in the asymmetric cost case. Each player has a uniformly weighted prior.

Figure 1.2 presents the trajectories for the actual payoffs π_{it} achieved by each player *i* at each time period *t*. Once again, I assume that each player has a uniformly weighted prior. The large variation from period to period is a result of players' randomly selecting one strategy to play from their mixed strategy vectors. In the symmetric case, each player *i*'s per-period payoff hovers close to the symmetric Nash equilibrium payoff of π_i^{NE} = 37.5. On average, however, both players do slightly worse than the Nash equilibrium, both averaging payoffs of 37.3 (s.d. = 2.96 for player 1 and s.d. = 2.87 for player 2).[14] In the asymmetric case, the vector of players' per-period payoffs is once again close to the Nash equilibrium payoff vector π^{NE} = (21, 51). However, player 1 slightly outperforms her Nash equilibrium, averaging a payoff of 21.16 (s.d. = 2.16), while player 2 underperforms, averaging a payoff of 50.34 (s.d. = 2.59). Thus, when costs

[14] The average and standard deviation for the payoffs are calculated as follows: means and standard deviations are first taken for all *T*=1000 time periods for one simulation, and then the values of the means and standard deviations are averaged over 20 simulations.

are asymmetric, the high-cost firm does better on average under logistic smooth fictitious play than in the Nash equilibrium, while the low-cost firm does worse on average.[15]



(a) (b)

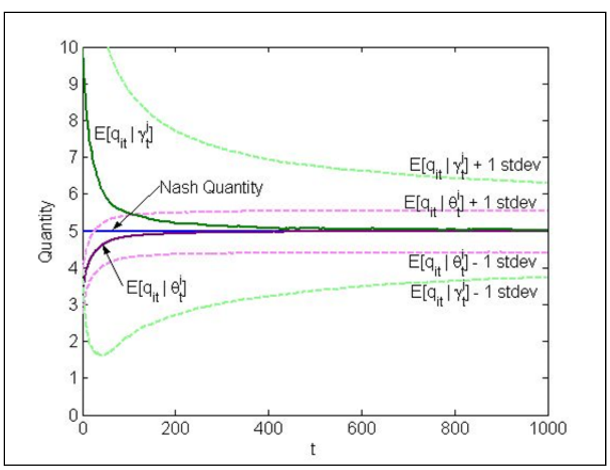
Figure 1.2. Actual payoffs achieved by each player as a function of time in the (a) symmetric and (b) asymmetric cases in the absence of Nature. Each player has a uniformly weighted prior.

Much of the variation in the achieved payoff arises from the fact at each time t, each player i randomly selects one strategy q_{it} to play from his time-t mixed strategy vector θ_t^i. By taking the mean over these vectors at each time t, I can eliminate this variation and gain a clearer picture of the dynamics of each player's strategy. Figure 1.3 presents the evolution of the expected per-period quantities, where expectations are taken at each time t either over players' mixed strategies or over opponents' assessments at time t, values corresponding to expressions (1) and (2), respectively. As before, each player has a uniformly weighted prior. Figures 1.3(a) and 1.3(b) present the both mean of player 1's mixed strategy (i.e., $E[q_1 \mid \theta_t^1]$) and the mean of player 2's assessment of what player 1 will play (i.e., $E[q_i \mid \gamma_t^j]$) for the symmetric- and

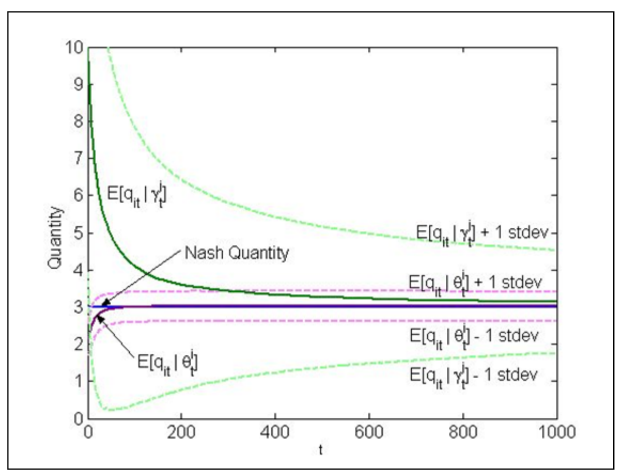
[15] This qualitative result is robust across the two sets of cost parameters I analyzed.

asymmetric-cost cases, respectively. Figure 1.3(c) gives the mean of player 2's mixed strategy and the mean of player 1's assessment of player 2 in the asymmetric case.

For both the symmetric and asymmetric cost cases, the mean of player 2's assessment is initially very high and asymptotically approaches the Nash equilibrium. As explained above, this is a result of the uniformly weighted prior. Initially, player 2 expects player 1 to play an average strategy of 10. Similarly, player 1 expects player 2 to play an average strategy of 10, and consequently player 1's mixed strategy initially has a very low mean, which rises asymptotically to the Nash equilibrium. It is interesting to note that in the asymmetric case, the mean over player 1's chosen mixed strategy slightly overshoots the Nash equilibrium and then trends back down towards it. Figure 1.3 also provides standard deviations over player 1's mixed strategy and player 2's assessment. Note that the standard deviation of player 2's assessment is much higher than the standard deviation of player 1's mixed strategy, indicating player 2's relative uncertainty about what player 1 is doing.[16]



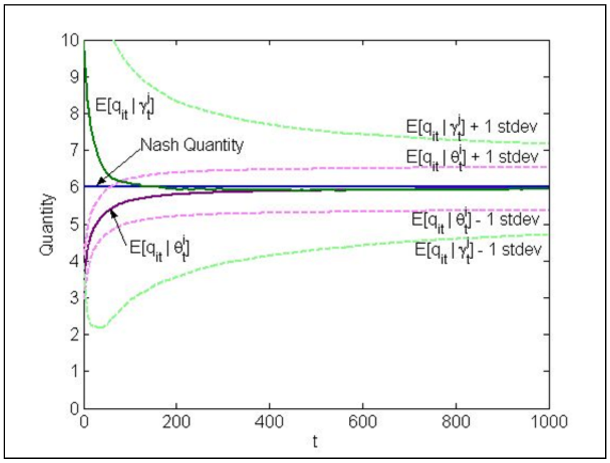
(a)



(b)

---

[16] Although the results presented in these figures are the outcome of one particular simulation, in general the variation in the values for the expected quantities across simulations is small.



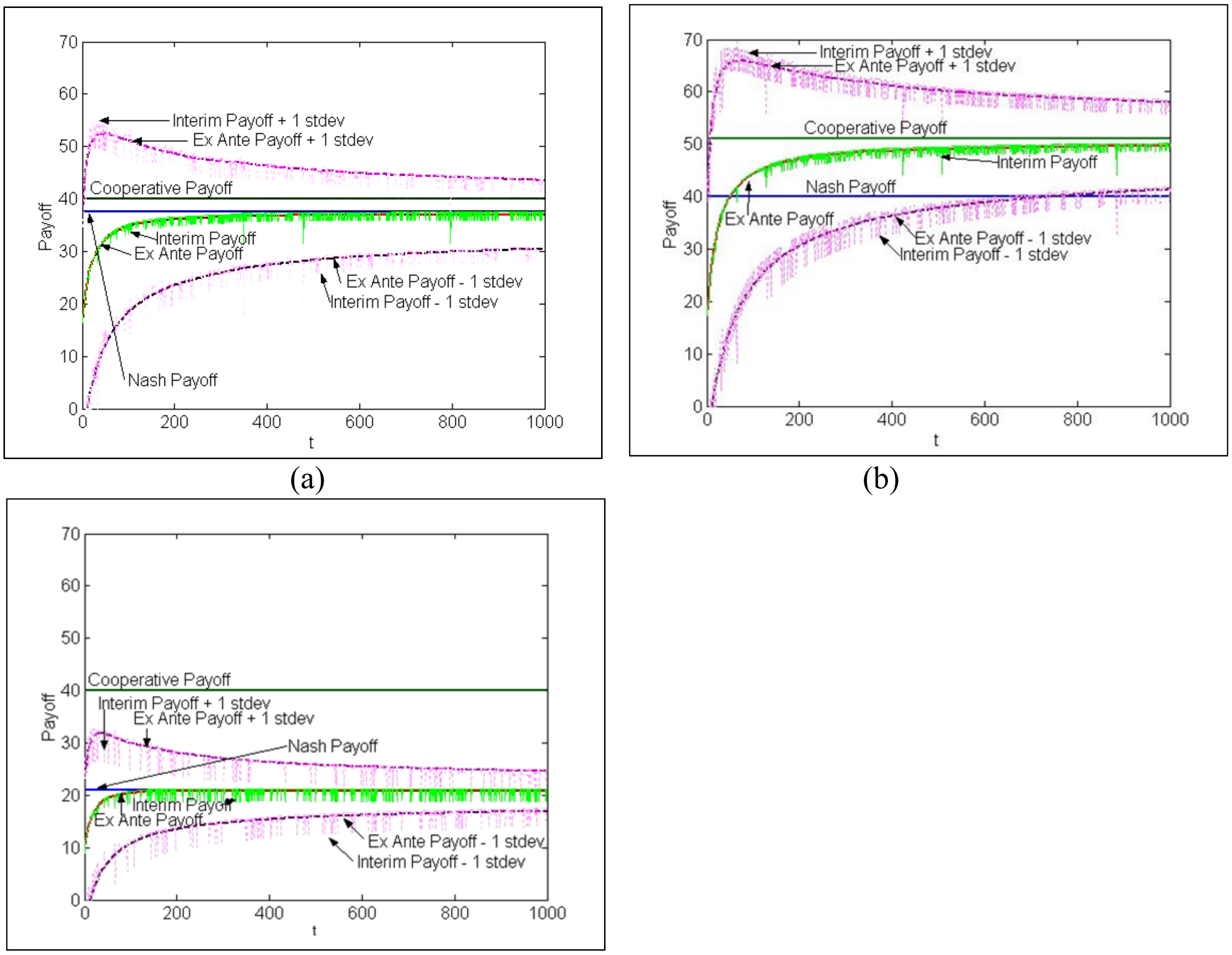
(c)

Figure 1.3. Means and variances of quantities, as taken over players' time-*t* mixed strategies and opponent's time-*t* assessments in the absence of Nature in the (a) symmetric case and the asymmetric case for (b) the higher-cost player 1 and (c) the lower-cost player 2 as a function of time. Each player has a uniformly weighted prior.

Just as an examination of the expected per-period quantity instead of the mixed strategy vector can elucidate some of the dynamics underlying play, analyzing expected payoffs can similarly eliminate some of the variation present in the trajectories of players' achieved payoffs in Figure 1.2. Figure 1.4 presents the evolutions of players' ex ante and interim expected payoffs, corresponding to expressions (3) and (4), respectively. Figures 1.4(a) and 1.4(b) depict these quantities for player 1. The interim payoff has a large variance from period to period because it is calculated after player 1 has randomly selected a strategy from his mixed strategy. In the symmetric case, depicted in Figure 1.4(a), both the ex ante and interim expected payoffs asymptote to the Nash equilibrium payoff, but remain slightly below it. In the asymmetric cost case, the high-cost player 1 eventually does better than she would have in Nash equilibrium while the low-cost player 2 eventually achieves approximately his Nash equilibrium payoff.[17] For all cases, on average, the interim expected payoff is below the ex ante expected payoff. Figure 1.4 also presents standard deviations for the ex ante and interim expected payoffs; in

[17] In the alternative set of cost parameters I tried, the high-cost player 1 eventually achieves approximately her Nash equilibrium payoff in the long run while the low-cost player 2 does worse than his Nash equilibrium payoff.

general they seem roughly equal.[18] Thus, while players in the symmetric cost case do slightly worse than in Nash equilibrium in the long run, the high-cost player 1 in the asymmetric cost case does better in the long run under stochastic fictitious play than she would in Nash equilibrium.



(c)

Figure 1.4. Means and variances of ex ante and interim payoff in the absence of Nature in the (a) symmetric case and the asymmetric case for (b) the higher-cost player 1 and (c) the lower-cost player 2 as a function of time. Each player has a uniformly weighted prior.

Having shown that expected quantity and expected payoff seem to converge to the Nash equilibrium, I now test whether this is indeed the case. First, I examine whether or not the mixed

[18] As before, although the results presented in these figures are the outcome of one particular simulation, in general the variation in the values for the ex ante and interim payoffs across simulations is small.

strategies do converge and the speed at which they converge. Figure 1.5 gives a measure of the convergence of smooth fictitious play when priors are uniformly weighted. As explained above, I define how close to steady-state player i is at time t as the maximum Euclidean distance between player i's mixed strategy vector θ_t^i at times $m, n \geq t$. Indeed, the mixed strategies do converge: the Euclidean distance asymptotes to zero. In the symmetric case, Figure 1.5(a), both players converge at approximately the same rate. In the asymmetric case, Figure 1.5(b), the player with higher costs, player 1, appears to converge more quickly.

Now that I have established that the mixed strategies do indeed converge, the next question I hope to answer is whether they converge to the Nash equilibrium. Figures 1.5(c) and 1.5(d) depict the Euclidean distance between player i's mixed strategy vector θ_t^i and the Nash equilibrium. In the symmetric case, both players converge at about the same rate, but neither gets very close to the Nash equilibrium. In the asymmetric case, player 1 again stabilizes more quickly. Furthermore, player 1 comes much closer to the Nash equilibrium than player 2 does. With uniformly-weighted priors, it is never the case that $d(X_t, X^{NE}) < \overline{d}$, where $\overline{d} = 0.01$; thus neither player converges to the Nash equilibrium. At time T=1000, the distance to the Nash equilibrium is 0.21 in the symmetric case. In the asymmetric case, the distance to Nash equilibrium is 0.39 for the higher cost player and 0.53 for the lower cost player.

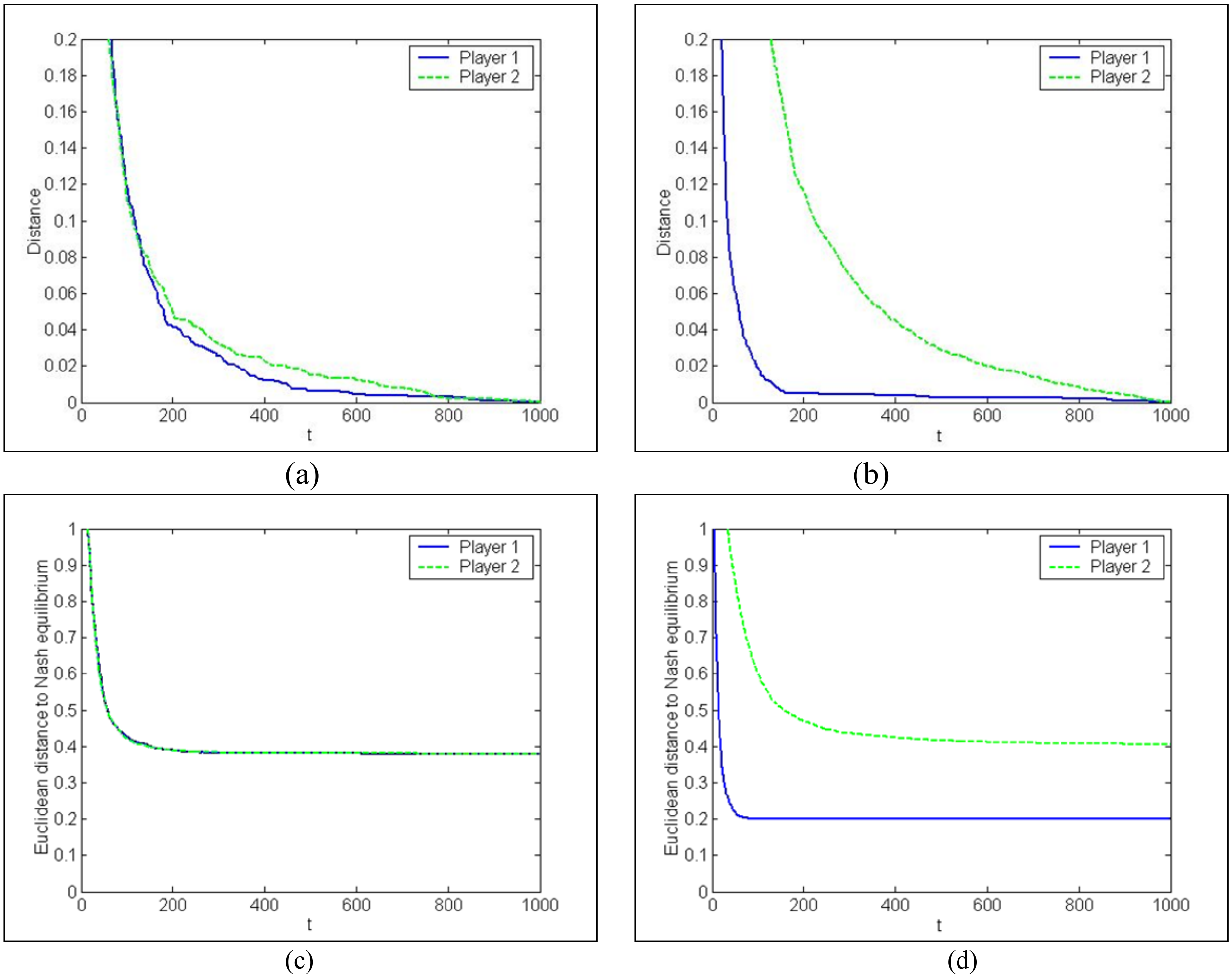


Figure 1.5. Maximum Euclidean distance between player i's mixed strategy vector θ_t^i in periods $m, n \geq t$ in the absence of Nature in the (a) symmetric and (b) asymmetric cases as a function of time. Distance between player i's mixed strategy vector and the Nash equilibrium in the (c) symmetric and (b) asymmetric cases. Each player has a uniformly weighted prior.

Because the players' prior beliefs are responsible for much of the behavior observed in the early rounds of play, I now examine how the mean and the spread of the priors affect the convergence properties. First, I examine how my results may change if instead of a uniformly weighted prior, each player i's had a prior that concentrated all the weight on a single strategy: $\kappa_0^i = (0, 0, \ldots, 21, 0, 0, \ldots, 0)$. I call such a prior a *concentrated prior*.

Figure 1.6 repeats the analyses in Figure 1.5, but with concentrated priors that place all the weight on quantity 9. The figures show that in both the symmetric and asymmetric cases, the form of the prior affects the speed of convergence but not its asymptotic behavior. Even with concentrated priors, each player's play still converges to a steady state mixed strategy vector. With concentrated priors, just as with uniformly weighted priors, the distance to the Nash equilibrium converges to 0.21 in the symmetric case and 0.39 and 0.53 in the asymmetric case.

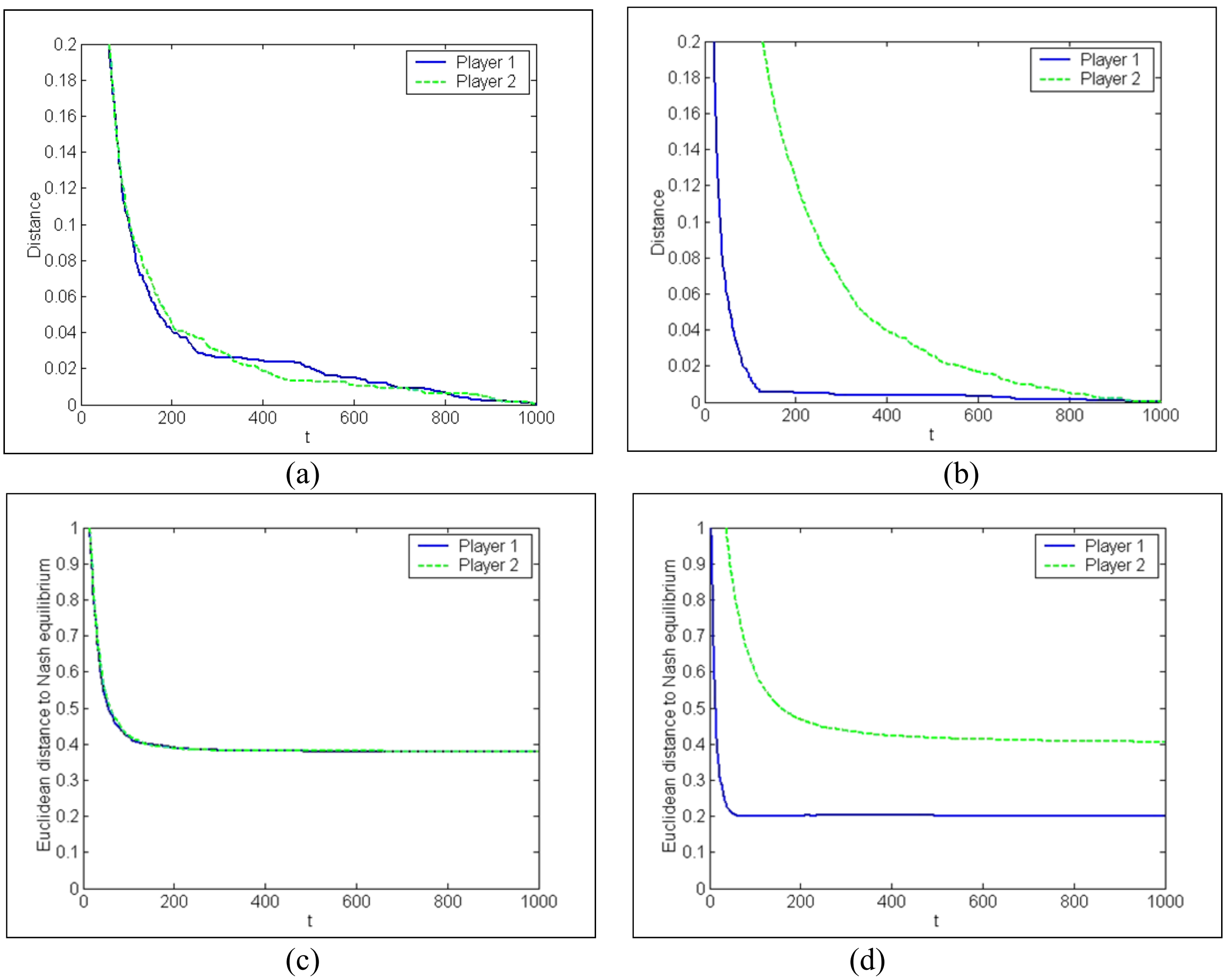


Figure 1.6. Maximum Euclidean distance between player *i*'s mixed strategy vector θ_t^i in periods $m, n \geq t$ in the absence of Nature in the (a) symmetric and (b) asymmetric cases as a function of time. Distance between player *i*'s mixed strategy vector and the Nash equilibrium in the (c) symmetric and (b) asymmetric cases. Each player *i* has a concentrated prior that places all the weight on the strategy $q_j = 9$.

I now examine the effect of varying the means of the concentrated priors on the mean quantity $E[q_i \mid \theta_t^i]$ of each player *i*'s mixed strategy. For each player, I allow the strategy with the entire weight of 21 to be either 4, 8, 12, or 16. Thus, I have 16 different combinations of initial priors. The phase portraits in Figure 1.7 are produced as follows. For each of these combinations of priors, I calculate each player *i*'s expected quantity over their mixed strategies $E[q_1 \mid \theta_t^1]$ and plot this as an ordered pair for each time *t*. Each trajectory thus corresponds to a different specification of the priors, and displays the evolution of the mixed strategy over T=1000 periods. The figure shows that in both cases, no matter what the prior, the players converge to a point close to the Nash equilibrium. In fact, the endpoints, corresponding to T=1000, appear to fall on a line. It is also interesting to note that many of the trajectories are not straight lines, indicating that players are not taking the most direct route to their endpoints. Notice that in the asymmetric case player 2's quantity never gets very far above her Nash equilibrium quantity.

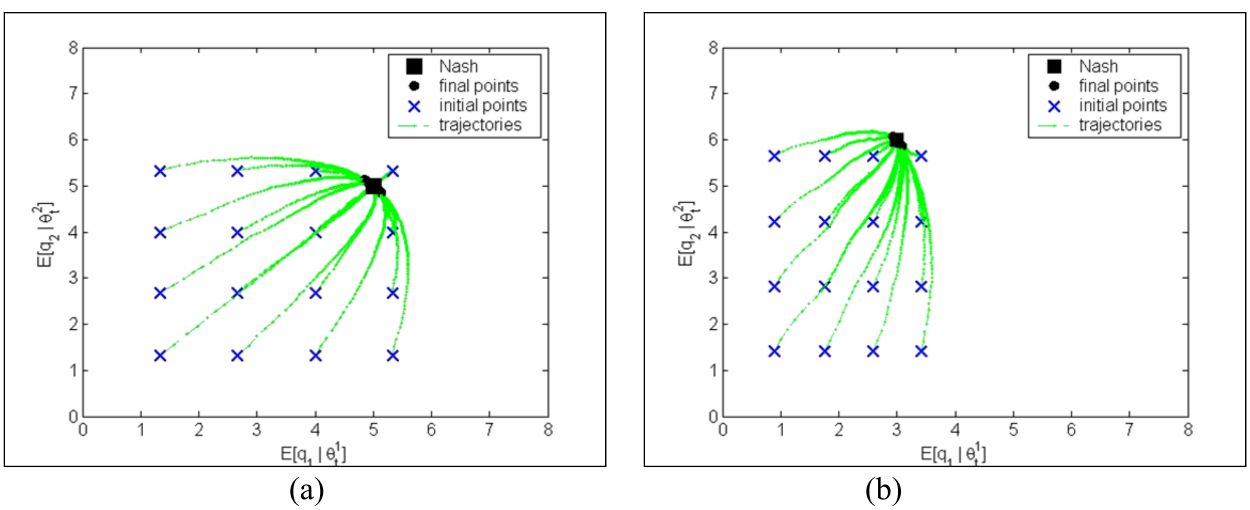
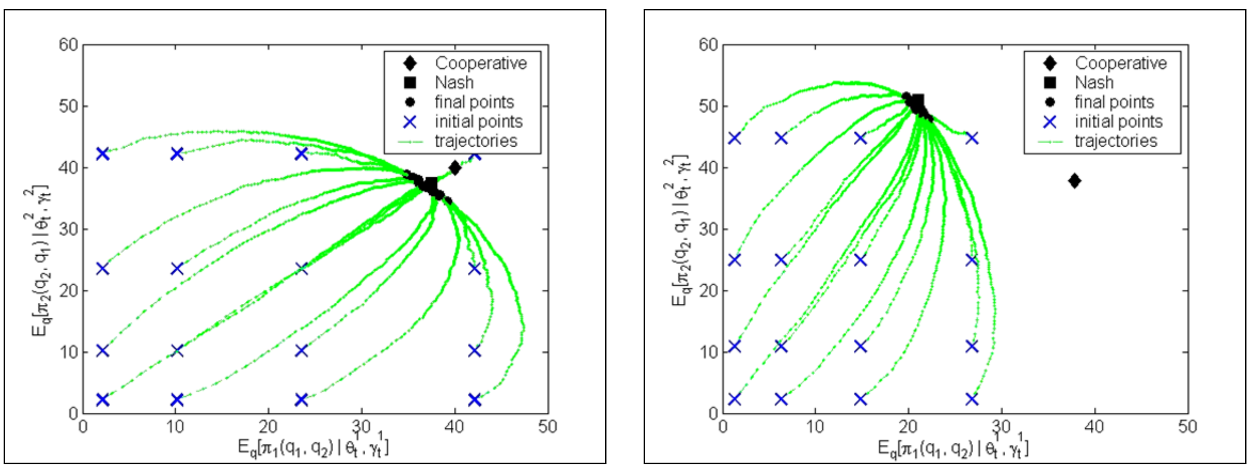


Figure 1.7. Phase portraits of expected quantity show the effect of varying (concentrated) priors in the absence of Nature in the (a) symmetric and (b) asymmetric cases.

While Figure 1.7 shows phase portraits of expected quantity, Figure 1.8 shows phase portraits of ex ante expected payoff. For comparison, the payoffs from the Nash and cooperative equilibria are plotted as benchmarks. Once again, no matter the initial prior, the payoffs converge close to the Nash equilibrium payoffs in both cases. Again, the endpoints, corresponding to T=1000, appear to fall on a line. In this case, however, the Nash equilibrium appears to be slightly above the line. Thus, in the steady-state outcome of logistic smooth fictitious play, the players are worse off than they would be in a Nash equilibrium.



(a) (b)

Figure 1.8. Phase portraits of ex ante expected payoff in the absence of Nature in the (a) symmetric and (b) asymmetric cases.

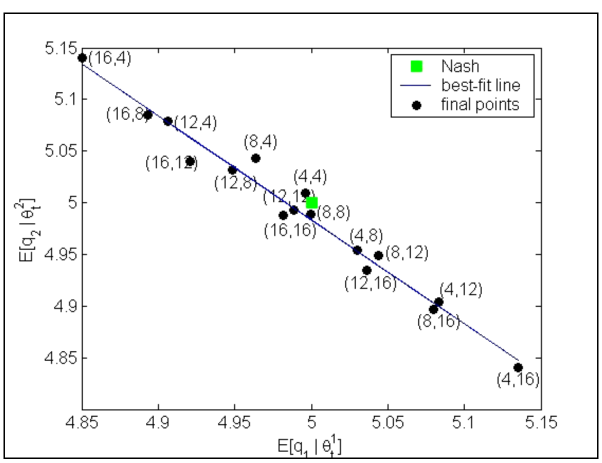
As noted above, the final points of the trajectories of expected quantity shown in Figure 1.7 seem to form a line, as do the final points of the trajectories of ex anted expected payoffs in Figure 1.8. Figure 1.9 shows only these final points and their best-fit line for both the expected quantity and for the ex ante expected payoff. As seen in Figure 1.6, each of the final points represents the long-run steady state reached by the players.

Several features of the results in Figure 1.9 should be noted. The first feature is the linear pattern of the final points. In the symmetric case, the slope of the best-fit line, which lies below the Nash equilibrium, is approximately -1.01 (s.e. = 3e-6). Thus, varying the prior appears only to affect the distribution of production between the two firms, but not the total expected quantity produced, and this total expected quantity is weakly less than that which arises in the Nash equilibrium. In the asymmetric case, the slope of the line, which again lies below the Nash equilibrium, is -1.58 (s.e. = 3e-6). Thus, a weighted sum of the expected quantities, where the higher cost player 1 is given a greater weight, is relatively constant across different priors. Similar statements can be made about the payoffs as well: that is, the sum of the payoffs is robust to the priors but lower than the sum of the Nash equilibrium payoffs in the symmetric cost case, and a weighted sum of the payoffs is robust to the priors but lower than the weighted sum of the Nash equilibrium payoffs in the asymmetric cost case.

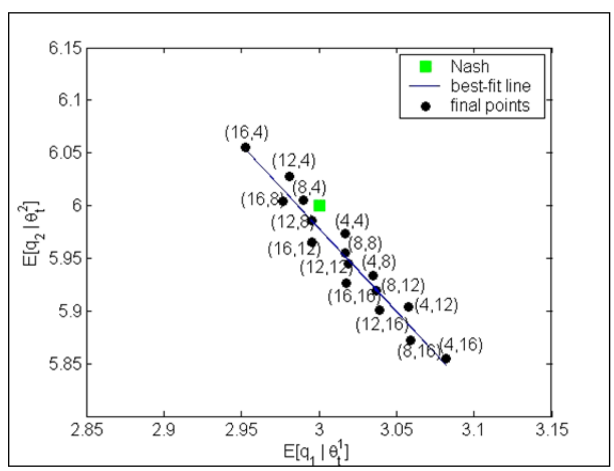
A second feature of Figure 1.9 to note regards how each player performs relative to his Nash equilibrium across different priors. In the symmetric case, the final points are distributed fairly evenly about the Nash equilibrium along the best-fit line. This implies that the number of priors for which player 1 does better than the Nash equilibrium is approximately equal to the number of priors for which player 2 does better than the Nash equilibrium. In the asymmetric case, on the other hand, most of the endpoints lie below the Nash equilibrium, implying that the number of priors for which player 1 does better than the Nash equilibrium is larger than the number of priors for which player 2 does better than the Nash equilibrium. This seems to confirm the earlier observation that the higher cost player usually outperforms her Nash equilibrium in the asymmetric case.

A third important feature of Figure 1.9 regards convergence. Note that Figures 1.9(a) and (b) show that there are several combinations of priors ((4,4), (8,8), (12, 12), and (16, 16) in the symmetric case, and (8,4), (12, 8) and (16,12) in the asymmetric case) that lead to steady state expected quantities very close to the Nash equilibrium (within a Euclidean distance of 0.01). However, convergence as earlier defined requires that the players' mixed strategy vectors, not the expectations over these vectors, come within a Euclidean distance of 0.01 of the Nash equilibrium. This does not happen for any set of priors; under no combination of priors do the players mixed strategy vectors converge to the Nash equilibrium.

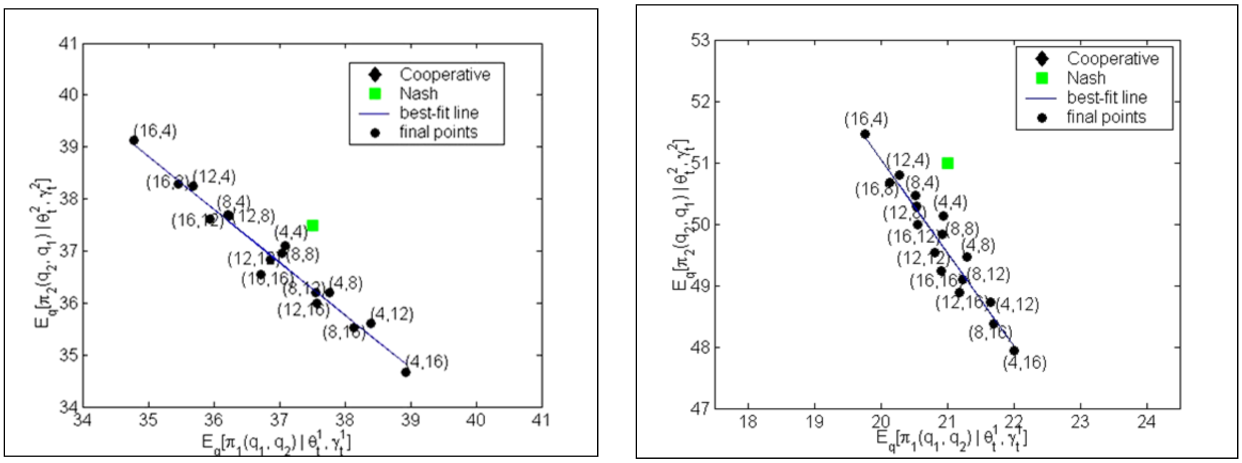
A fourth important feature of Figure 1.9 regards the effects of a player's prior on his long-run quantity and payoff. According to Figure 1.9, when the opponent's prior is held fixed, the lower the prior a player has over her opponent (i.e., the less she expects the other to produce), the more she will produce and the higher her per-period profit in the long run. There thus appears to be what I term the *overconfidence premium*: the worse off a player initially expects her opponent to be, the better off she herself will eventually be.



(a)



(b)

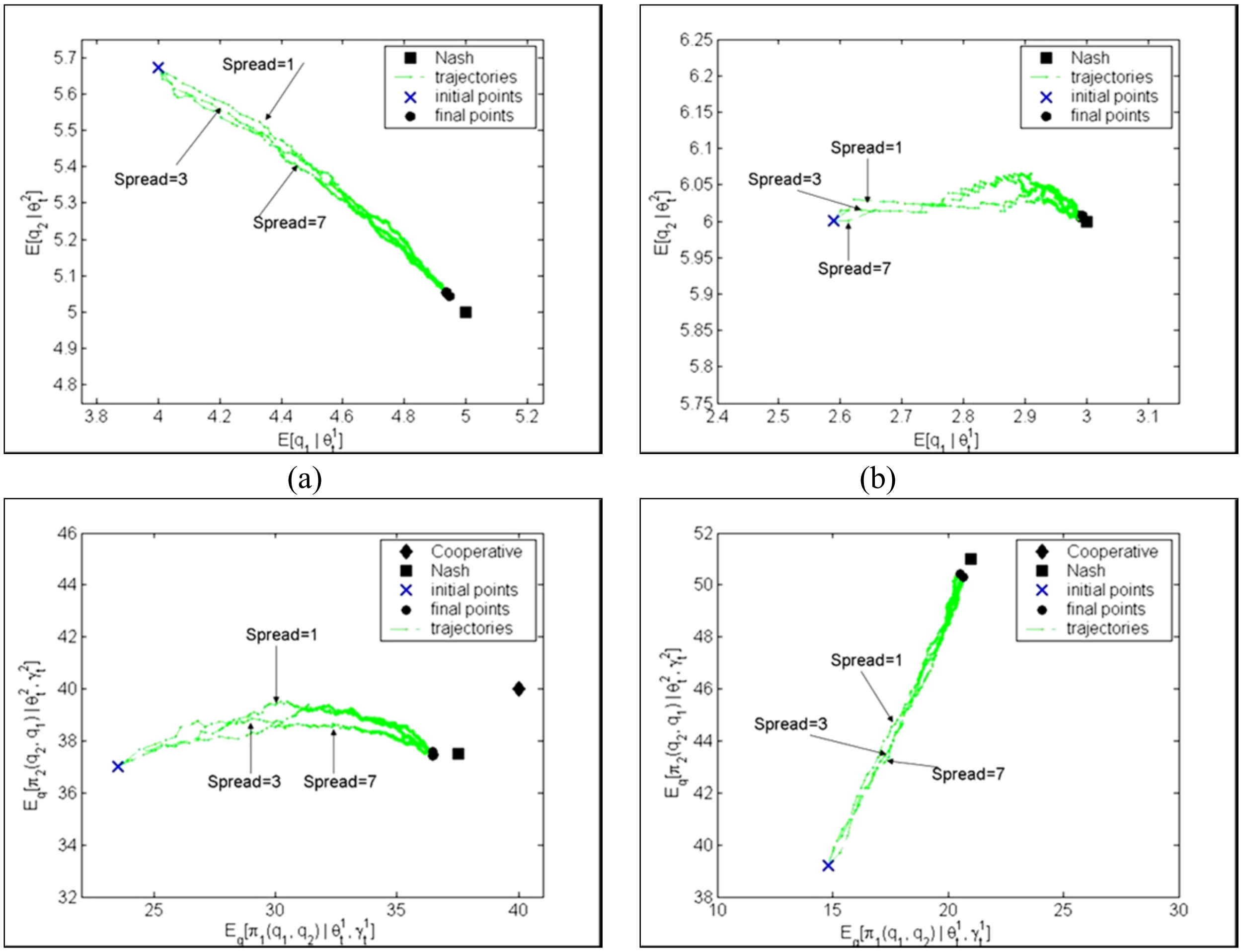


(c) (d)

Figure 1.9. Best fit lines of the endpoints of the trajectories of expected quantity shown in Figure 1.7 in the absence of Nature in the (a) symmetric and (b) asymmetric cases. Plots (c) and (d) give similar best fit lines for the trajectories of ex ante expected payoff shown in Figure 1.8.

Having seen the effect of varying the mean of each player's prior on the learning dynamics, I now fix the mean and vary the spread. Figure 1.10 shows the effect of spread in the prior. I fix player 2's prior, with all weight on one strategy (q_1 = 10).[19] Thus, player 2's prior looks like κ_0^2= (0, …, 0, 21, 0, … , 0). I vary player 1's prior, keeping its mean the same (also producing a quantity of 10), but spreading its weight over 1, 3, or 7 strategies. Thus, player 1's prior looks like one of κ_0^1= (0, …, 0, 21, 0, …, 0), (0, …, 0, 7, 7, 7, 0, …0), or (0,…, 0, 3, 3, 3, 3, 3, 3, 3, 0, …, 0). As I see in Figure 1.10, spreading the prior in this manner does affect the trajectory of expected quantities but does not alter the initial or final points in either the symmetric or asymmetric case. Each trajectory has the exact same starting point, while their final points vary just slightly. This variation decreases as the number of time steps increases. The same result arises if I plot phase portraits of the ex ante payoffs.

[19] I chose to concentrate the prior on the mean pure strategy in the strategy set both because it did not correspond to any Nash equilibrium, thus ensuring that the results would be non-trivial, and also so that varying the spread would be straightforward.

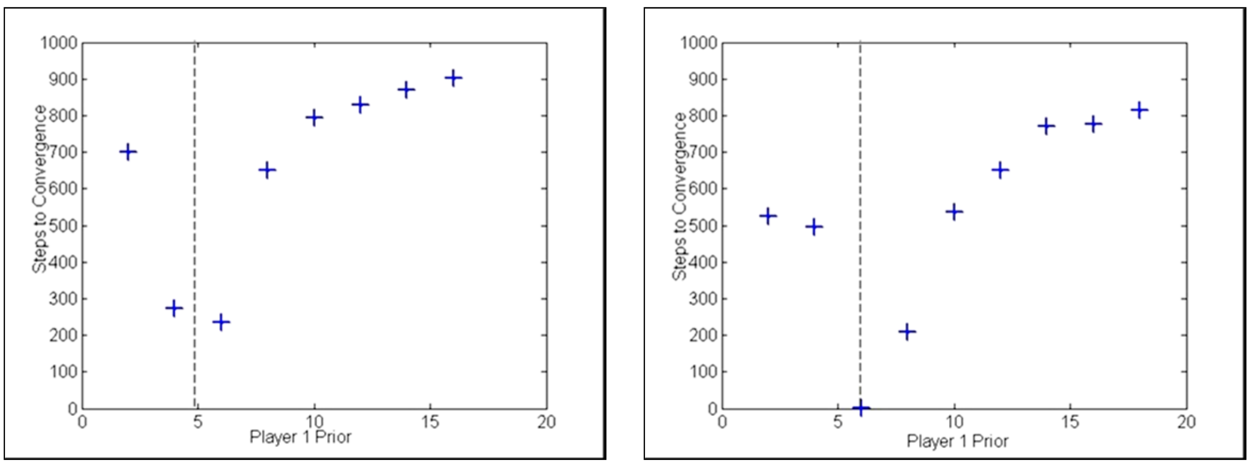


(a) (b)

(c) (d)

Figure 1.10. The effects of varying the spread of player 1's prior around the same mean ($q_2 = 10$) in the absence of Nature on the trajectories of expected quantity in the (a) symmetric and (b) asymmetric cases, and on the trajectories of ex ante payoff in the (c) symmetric and (d) asymmetric cases.

I next examine the effect of varying each player's prior on the rate of convergence. Figure 1.11 shows the effect of different priors on the speed of convergence of the mixed strategy for player 1. I hold player 2's prior fixed, with all weight on player 1's Nash equilibrium strategy (i.e., $q_1 = q_1^{NE} = 5$ for the symmetric cost case and $q_1 = q_1^{NE} = 3$ for the asymmetric cost case). I then vary player 1's prior, keeping all weight on one strategy, but varying that strategy between 2, 4, 6, 8, 10, 12, 14, 16, and 18. Player 2's Nash equilibrium strategy is indicated by the vertical dashed line.[20] In both cases, the time to convergence is minimized when player 1's prior puts all the weight on $q_2 = 6$. This is not surprising in the asymmetric case, Figure 1.9(b), because $q^{NE} = (3, 6)$ is the Nash equilibrium for that case. In the symmetric case, when player 1's prior puts all the weight on $q_2 = 6$, this is very close to player 2's Nash equilibrium quantity of $q_2^{NE} = 5$.



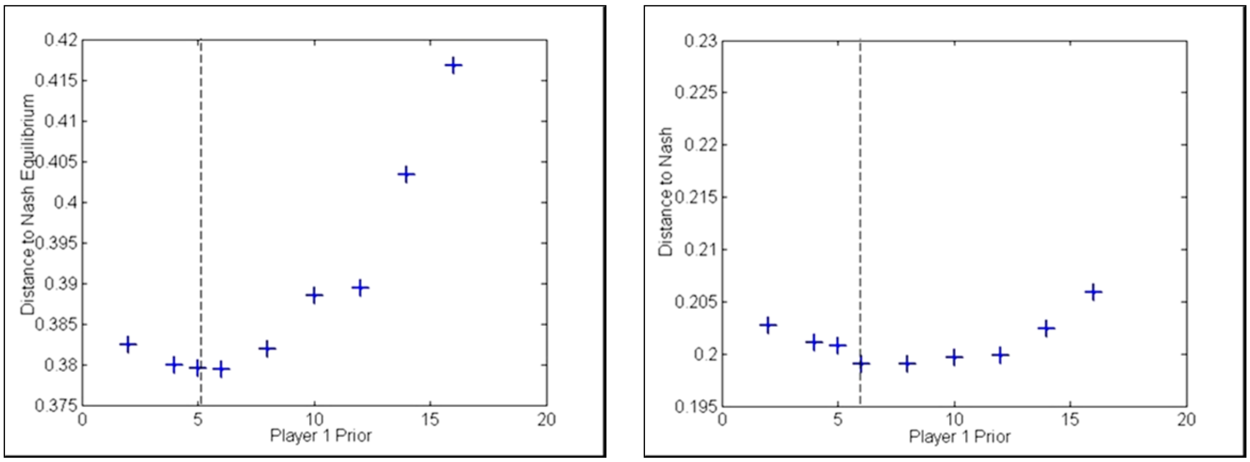
(a) (b)

Figure 1.11. The number of time steps until convergence in the absence of Nature in the (a) symmetric and (b) asymmetric cases. Player 1 has a concentrated prior. Player 2's Nash equilibrium strategy is indicated by the dashed line.

I now examine the effect of varying each player's prior on convergence to the Nash equilibrium. Figure 1.12 shows the effect of different priors on the final distance to the Nash equilibrium. Again, I hold player 2's prior fixed with all weight on player 1's Nash equilibrium strategy. I then vary player 1's prior as before. Player 2's Nash equilibrium strategy is again

[20] For the asymmetric cost case, one can also generate an analogous plot as a function of player 2's prior, holding player 1's prior constant at player 2's Nash equilibrium strategy.

shown by a dotted vertical line.[21] The distance between player 1's mixed strategy vector and his Nash equilibrium quantity at time T=1000 is smallest when player 1's prior is concentrated at a value close to player 2's Nash equilibrium quantity. The distance grows as player 1's prior gets further away from the Nash equilibrium quantity.



(a) (b)

Figure 1.12. Distance between player 1's mixed strategy vector and the Nash equilibrium at time T=1000 as a function of player 1's (concentrated) prior in the absence of Nature in the (a) symmetric and (b) asymmetric cases. Player 2's Nash equilibrium strategy is indicated by the dashed line.

Finally, I examine the effect of varying each player's prior on final-period ex ante payoff, as compared to the Nash equilibrium. Figure 1.13 shows the effect of different priors on the final ex ante payoff minus the Nash equilibrium payoff. Again, I hold player 2's prior fixed with all weight on player 1's Nash equilibrium strategy. I then vary player 1's prior as before. Player 2's Nash equilibrium strategy is indicated by a dotted vertical line.[22] The difference between player 1's ex ante payoff and the Nash equilibrium payoff at time T=1000 is largest when player 1's prior is smallest. The difference declines (and becomes negative) as player 1's prior grows.

[21] For the asymmetric cost case, one can also generate an analogous plot as a function of player 2's prior, holding player 1's prior constant at player 2's Nash equilibrium strategy.

[22] For the asymmetric cost case, one can also generate an analogous plot as a function of player 2's prior, holding player 1's prior constant at player 2's Nash equilibrium strategy.

When player 1's prior is smallest, he believes that player 2 will produce a small quantity. Thus, he will produce a large quantity, and reap the benefits of a larger payoff. This result confirms the overconfidence premium results from Figure 1.9: the worse off a player initially expects his opponent to be, the better off he himself will eventually be.

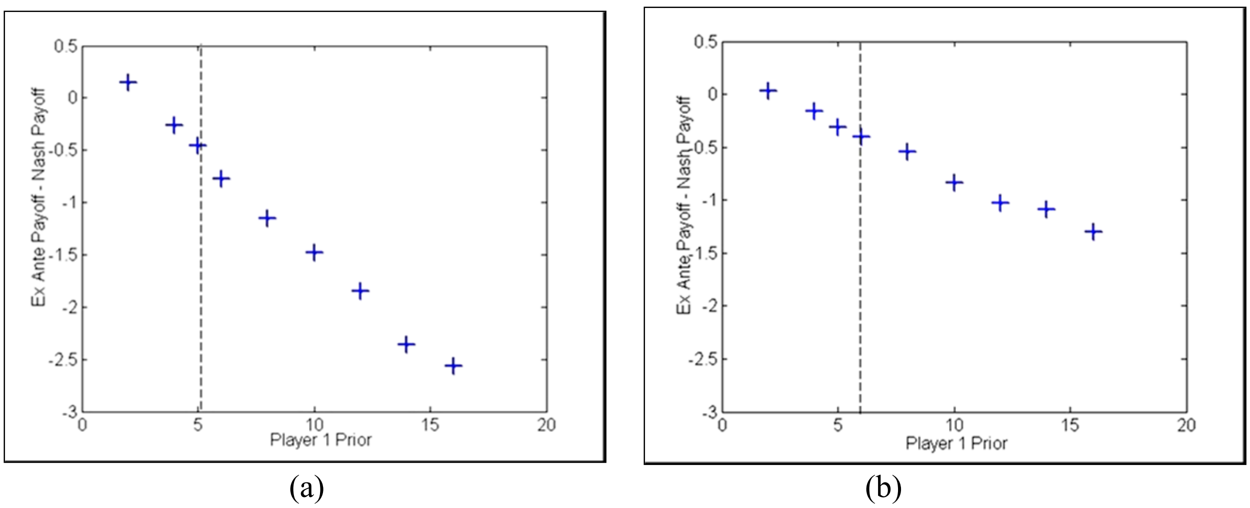


Figure 1.13. Difference between player 1's final-period ex ante payoff and the Nash equilibrium payoff at time T=1000 in the absence of Nature as a function of player 1's concentrated prior in the (a) symmetric and (b) asymmetric cases. Player 2's Nash equilibrium strategy is indicated by the dashed line.

In summary, the main results arising from my examination of the no Nature case are:[23]

1) In the symmetric case with uniformly weighted priors, players on average achieve a slightly smaller payoff than the Nash equilibrium payoff, both on average and in the long run.

[23] These qualitative results are robust across the two sets of cost parameters analyzed.

2) In the asymmetric case with uniformly weighted priors, the higher cost player outperforms her Nash equilibrium payoff both on average and in the long run, while the lower cost player underperforms his on average.

3) With either uniformly weighted priors or concentrated priors, both players' mixed strategy vectors converge to a steady state, but neither player's mixed strategy converges to the Nash equilibrium.

4) In the asymmetric case with uniformly weighted priors, the higher cost player's mixed strategy vector converges to steady state faster than that of the lower cost player. Furthermore, the higher cost player gets closer to the Nash equilibrium.

5) In the symmetric cost case, varying the priors affects the distribution of production and of payoffs between the two firms, but not either the total expected quantity produced nor the total payoff achieved, and both the total quantity and the total payoff are lower than they would be in equilibrium.

6) In the asymmetric cost case, varying the priors affects the distribution of production and of payoffs between the two firms, but not either the weighted sum of expected quantity produced nor the weighted sum of payoff achieved, and both the weighted sum quantity and the weighted sum payoff are lower than they would be in equilibrium.

7) Varying the spread of each player's prior while holding the mean fixed does not affect the long-run dynamics of play.

8) The distance between player 1's mixed strategy vector and his Nash equilibrium quantity at time T=1000 inversely related to the difference between the quantity at which player 1's prior is concentrated and player 2's Nash equilibrium quantity.

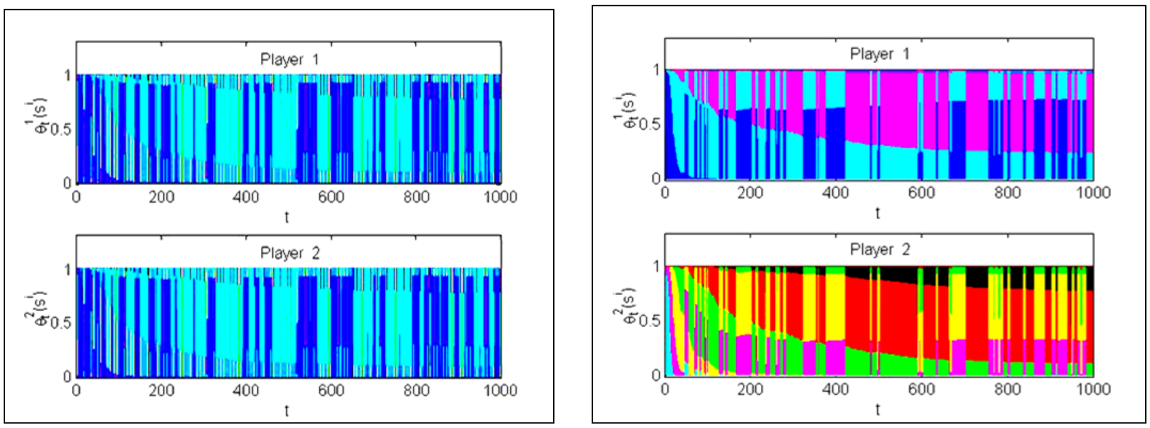
9) There is an *overconfidence premium*: the worse off a player initially expects her opponent to be, the better off she herself will eventually be.

5. Adding Nature as a Markov process

Having fully analyzed the no Nature benchmark case, I now examine the dynamics of stochastic fictitious play in the presence of Nature.

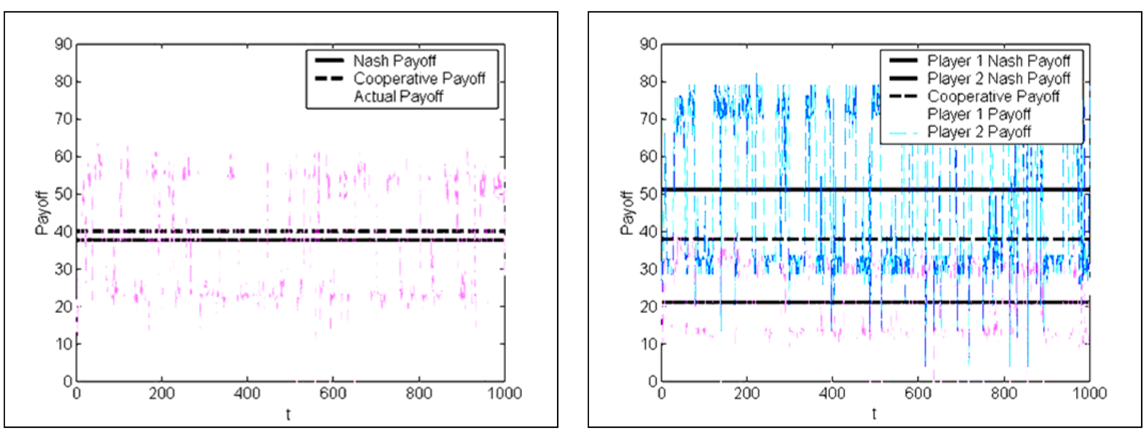
There are several key features of the dynamics that arise when Nature is a Markov process.[24] The first key feature is that the trajectories of play and payoffs are discontinuous. The discontinuities arise because players' assessments of Nature at each time *t* are conditional on value of the shock produced by Nature at time *t*-1. Conditional on any given value of the previous period's shock ε_{t-1}, the players' mixed strategies in Figure 2.1 evolve continuously; discontinuities arise, however, whenever the value of ε_{t-1} changes. The dynamics that arise when Nature is Markov thus pieces together the dynamics that arise when Nature evolves as each of three separate i.i.d. processes, one for each of the conditional distributions of ε_t. Similarly, trajectories for the actual payoffs achieved (Figure 2.2), for the means and variances of quantities as taken over players' time-*t* mixed strategies and opponent's time-*t* assessments (Figure 2.3), and for the means and variances of the ex ante and interim payoffs (Figure 2.4) are discontinuous, and can be viewed as an assemblage of pieces of three separate continuous trajectories.

[24] Note that because the Markov transition matrix generates a high degree of persistence in non-zero Nature shocks, the dynamics may vary from one simulation to the next. While the figures presented in this section plot the outcome arising from the realization of one particular sequence of shocks, my analysis focuses on the qualitative features of the results that were robust across the two parameter sets I tried.



(a) (b)

Figure 2.1. Dynamics of players' mixed strategies with (a) symmetric and (b) asymmetric costs as a function of time when Nature is Markov. As a benchmark, the Nash equilibrium quantities are $q^{NE} = (5,5)$ in the symmetric cost case and $q^{NE} = (3,6)$ in the asymmetric cost case. Each player has a uniformly weighted prior over the other.



(a) (b)

Figure 2.2. Actual payoffs achieved by each player as a function of time in the (a) symmetric and (b) asymmetric cases when Nature is Markov. Each player has a uniformly weighted prior over the other.

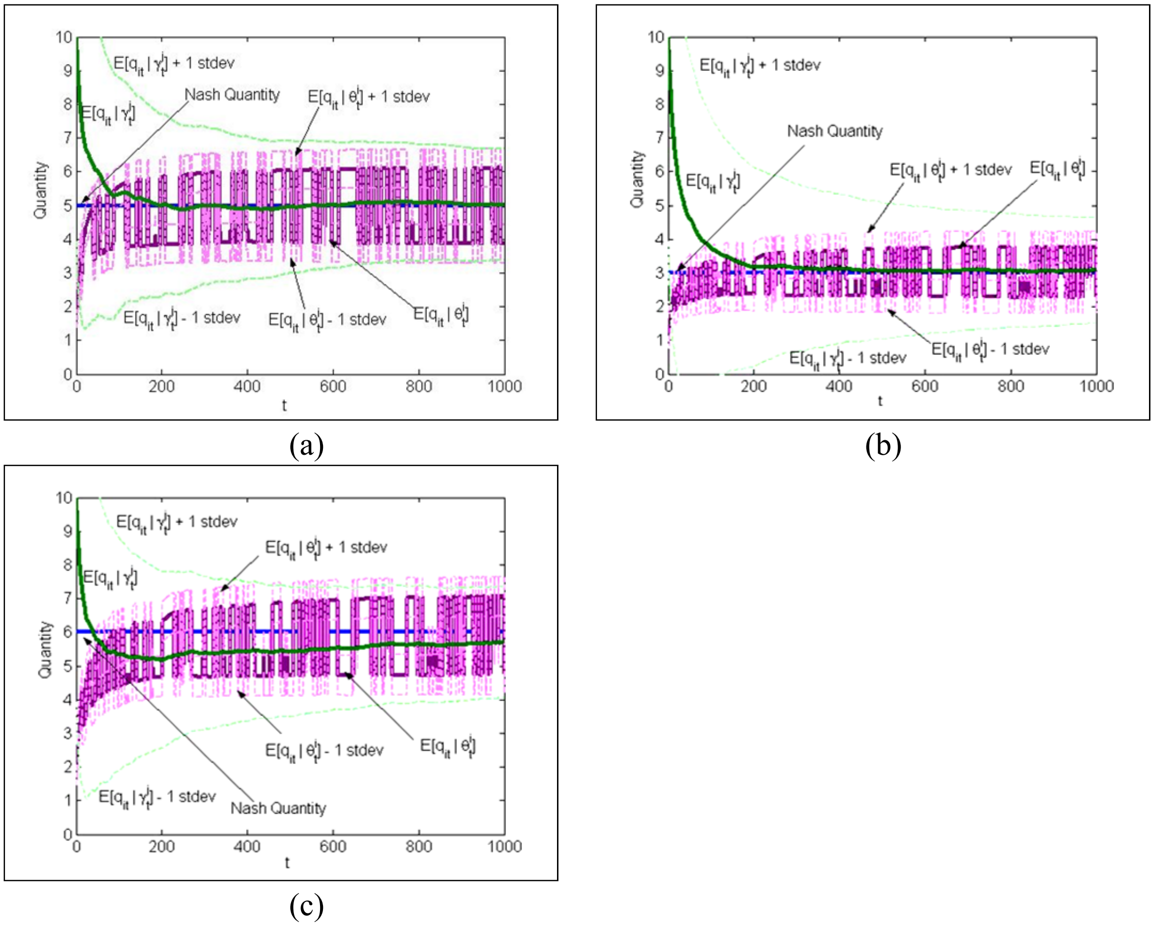
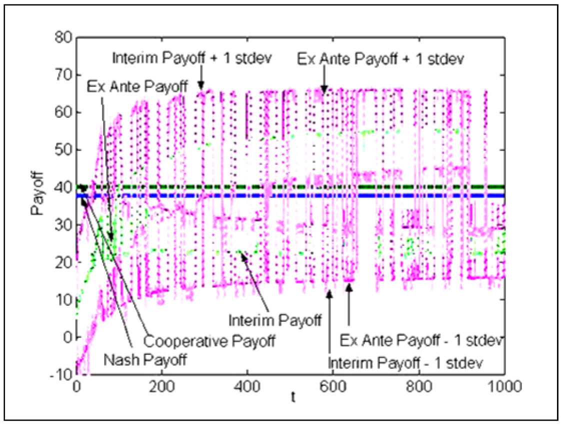
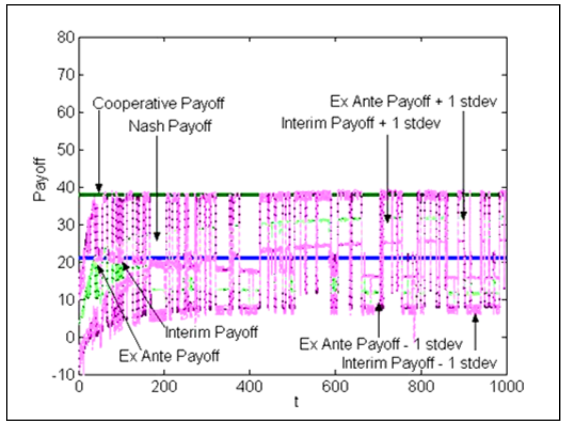


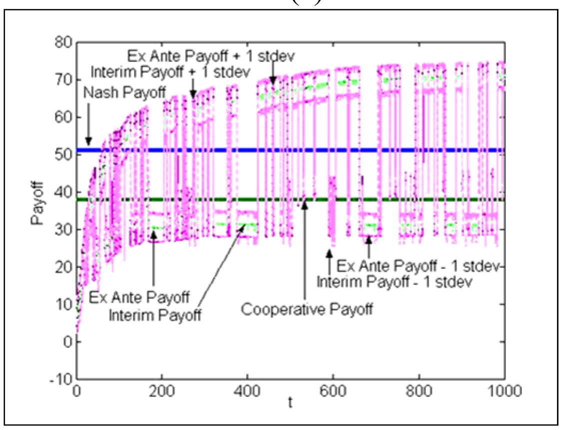
Figure 2.3. Means and variances of quantities, as taken over players' time-*t* mixed strategies and opponent's time-*t* assessments when Nature is Markov in the (a) symmetric case and the asymmetric case for (b) the higher-cost player 1 and (c) the lower-cost player 2 as a function of time. Each player has a uniformly weighted prior over the other.



(a)



(b)



(c)

Figure 2.4. Means and variances of ex ante and interim payoff when Nature is Markov in the (a) symmetric case and the asymmetric case for (b) the higher-cost player 1 and (c) the lower-cost player 2 as a function of time. Each player has a uniformly weighted prior over the other.

In addition to the discontinuous nature of the trajectories of play and payoffs, a second key feature of the dynamics that arise when Nature is Markov is the lack of convergence. Although play may converge conditional on any of the three values of ε_{t-1}, the overall dynamics, taken over all possible realizations of ε_{t-1}, does not. In other words, while each of the three separate i.i.d. Nature scenarios may lead to convergence, the assemblage of these three disjoint pieces does not. Thus, as seen in Figure 2.5, mixed strategies do not converge when players have uniformly weighted priors on each other, nor do they approach the Nash equilibrium.[25] Likewise, as seen in Figure 2.6, payoffs do not converge either. Similarly, play does not converge when priors are concentrated (Figure 2.11).[26]

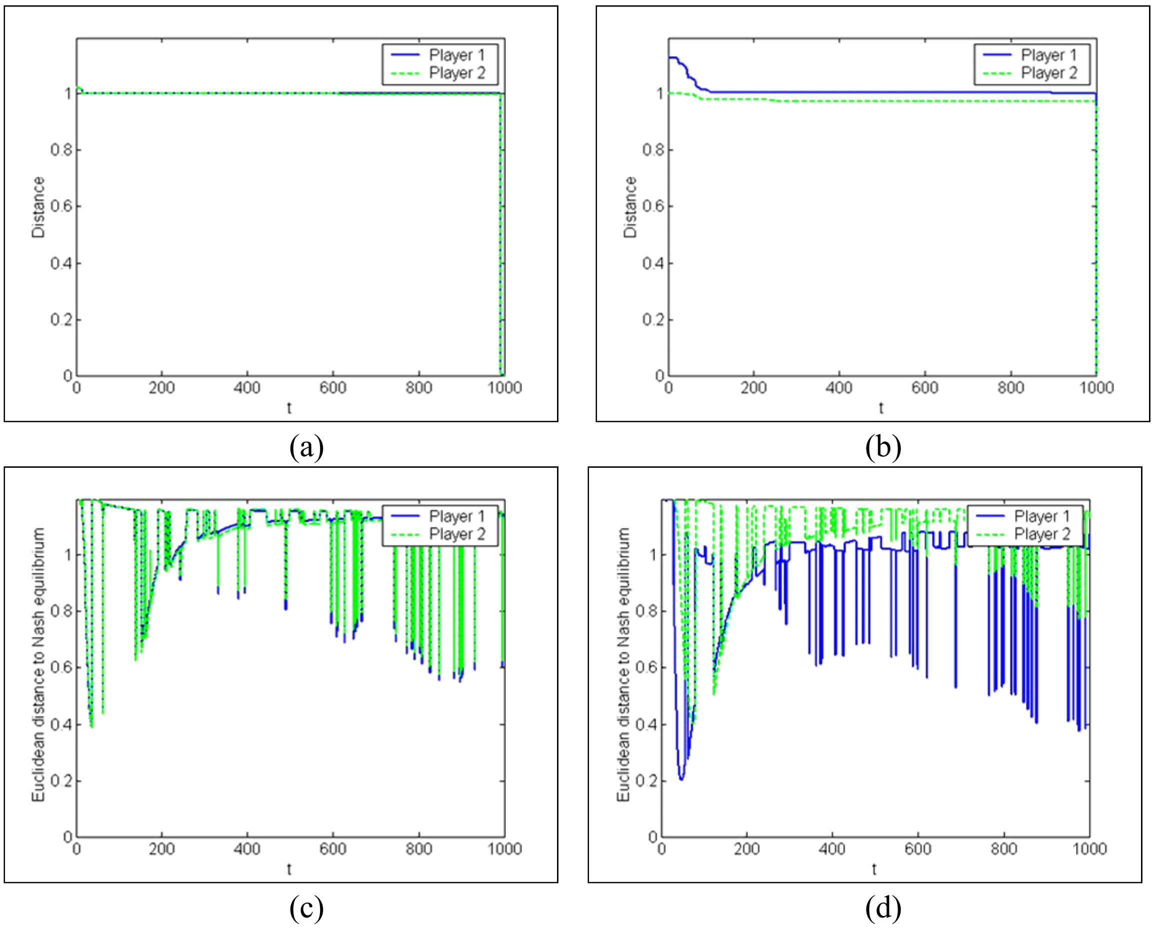
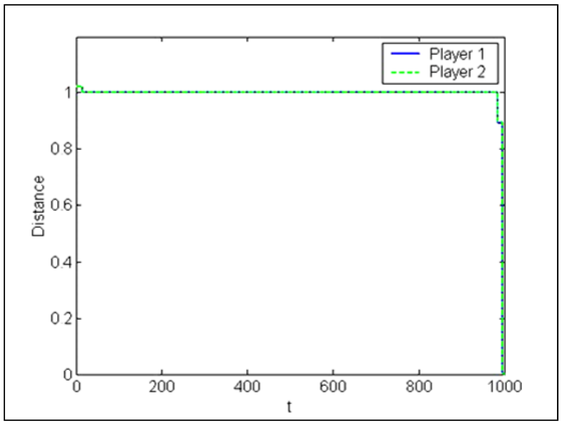


Figure 2.5. Maximum Euclidean distance between player *i*'s mixed strategy vector θ_t^i in periods $m, n \geq t$ when Nature is Markov in the (a) symmetric and (b) asymmetric cases as a function of time. Distance between player *i*'s mixed strategy vector and the Nash equilibrium in the (c) symmetric and (b) asymmetric cases. Each player has a uniformly weighted prior on the other.

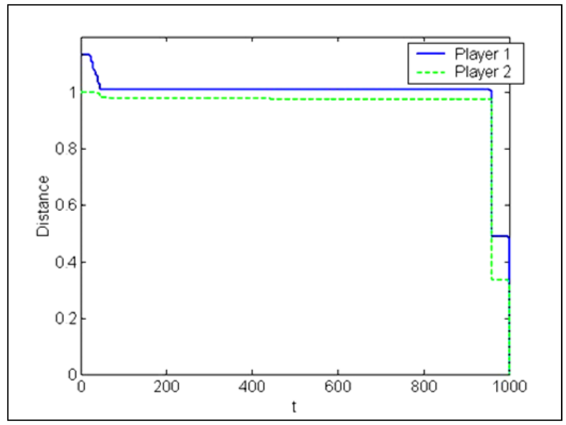
---

[25] In Figure 2.5, maximum Euclidean distance drops to 0 at t = 1000 not because convergence occurs, but because I truncate my simulations at T=1000. Because I calculate distance as $d(t) = \max[EuclideanDist(X_m, X_n)] \quad \forall m, n \in [\tau, T]$, d(1000)=0. An alternative way to calculate convergence would be to run the simulations for more than T time steps, say 1500 time steps, but then only calculate distance up to time T.

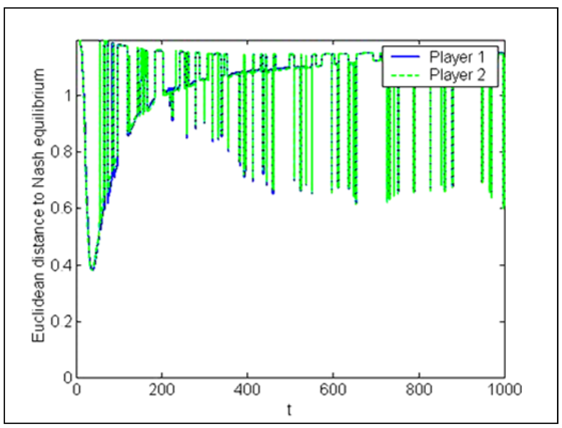
[26] To best enable comparisons between the no Nature and the Markov Nature cases, I chose to number the Markov figures according to the order of the analogous figures in the no Nature case rather in the order of their appearance in the text.



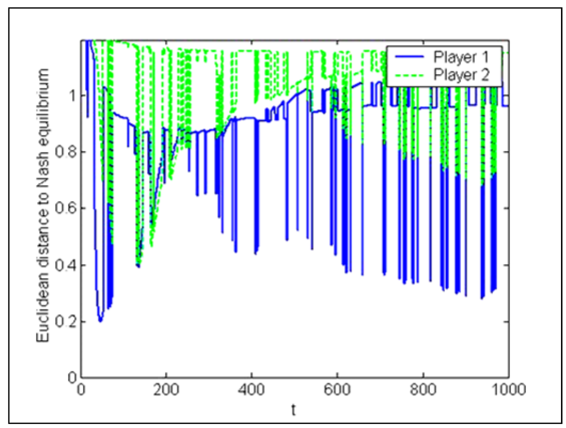
(a)



(b)

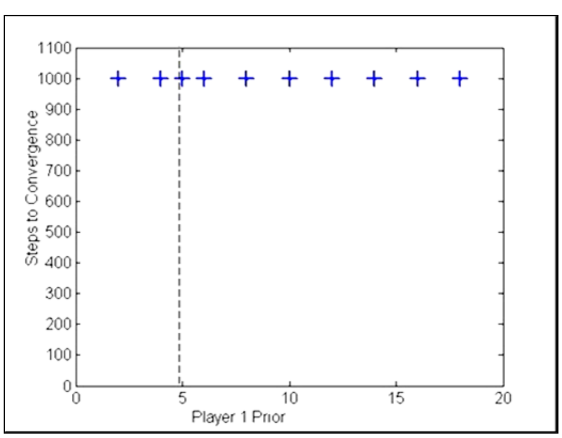


(c)

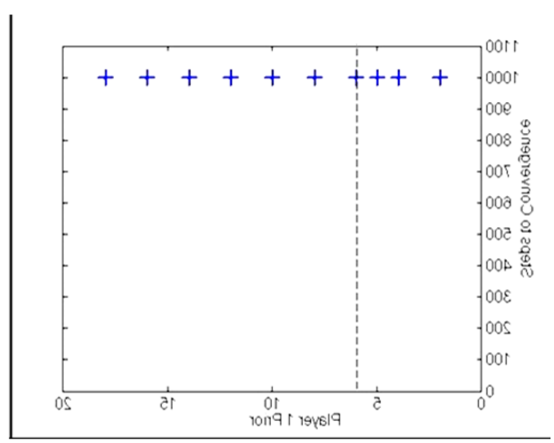


(d)

Figure 2.6. Maximum Euclidean distance between player *i*'s mixed strategy vector θ_t^i in periods $m, n \geq t$ when Nature is Markov in the (a) symmetric and (b) asymmetric cases as a function of time. Distance between player *i*'s mixed strategy vector and the Nash equilibrium in the (c) symmetric and (b) asymmetric cases. Each player *i* has a concentrated prior that places all the weight on the strategy $q_j = 9$.



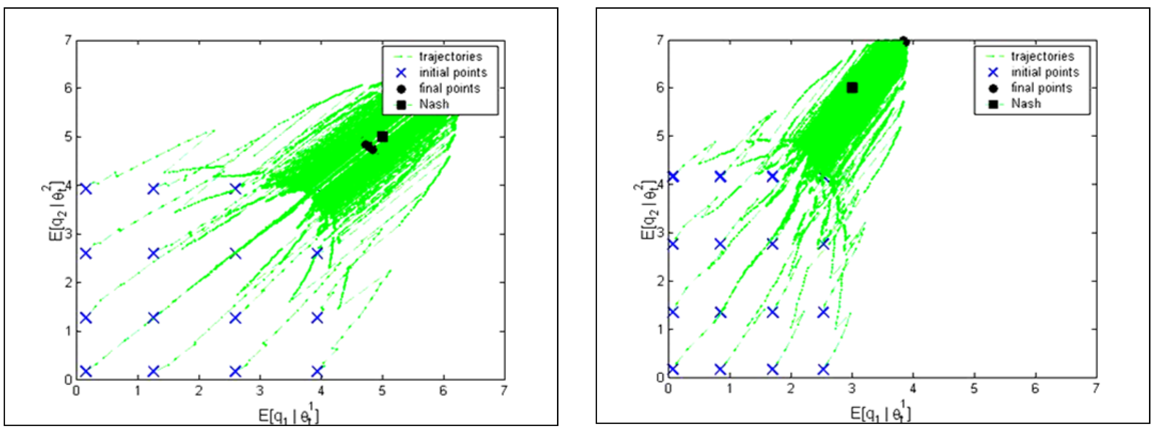
(a)



(b)

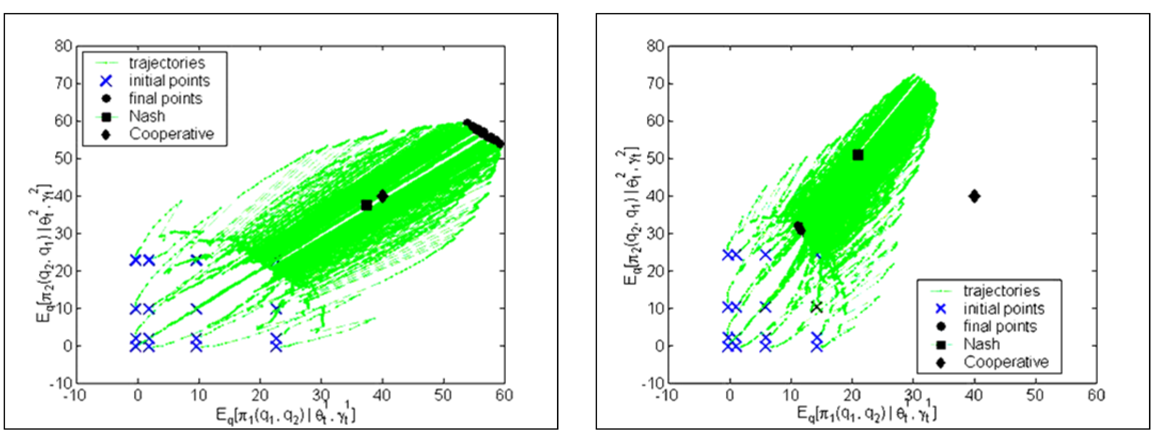
Figure 2.11. The number of time steps until convergence when Nature is Markov in the (a) symmetric and (b) asymmetric cases. Player 2's Nash equilibrium strategy is indicated by the dashed line.

A third key feature of the dynamics that arise when Nature is Markov is that while neither play nor payoffs converge, they appear to eventually enter an ergodic set, as seen in the phase portraits in Figures 2.7 and 2.8. Each of the trajectories in these phase portraits was generated from the same sequence of shocks. For the particular sequence of shocks used, $\varepsilon_{T-1}=\varepsilon_T=4$.



(a) (b)

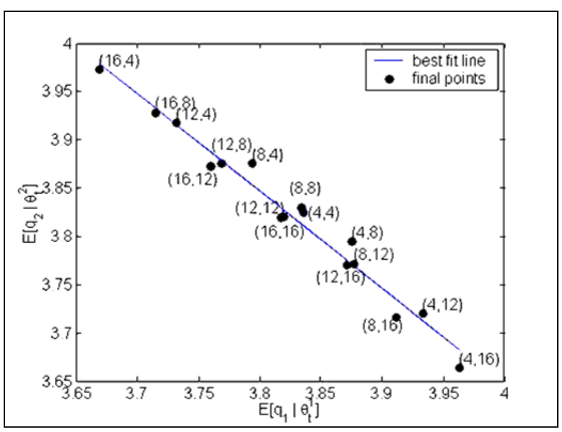
Figure 2.7. Phase portraits of expected quantity show the effect of varying priors over each other when Nature is Markov in the (a) symmetric and (b) asymmetric cases.



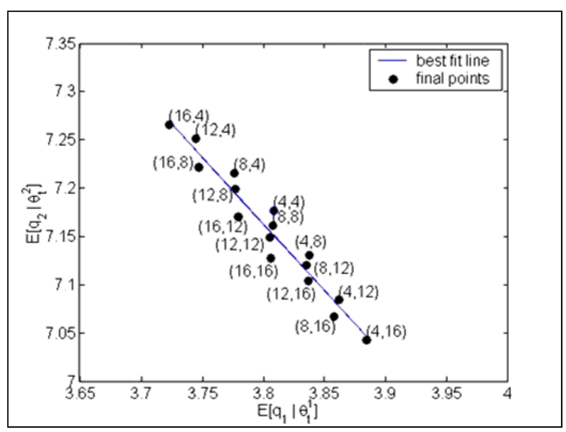
(a) (b)

Figure 2.8. Phase portraits of ex ante expected payoff when Nature is Markov in the (a) symmetric and (b) asymmetric cases.

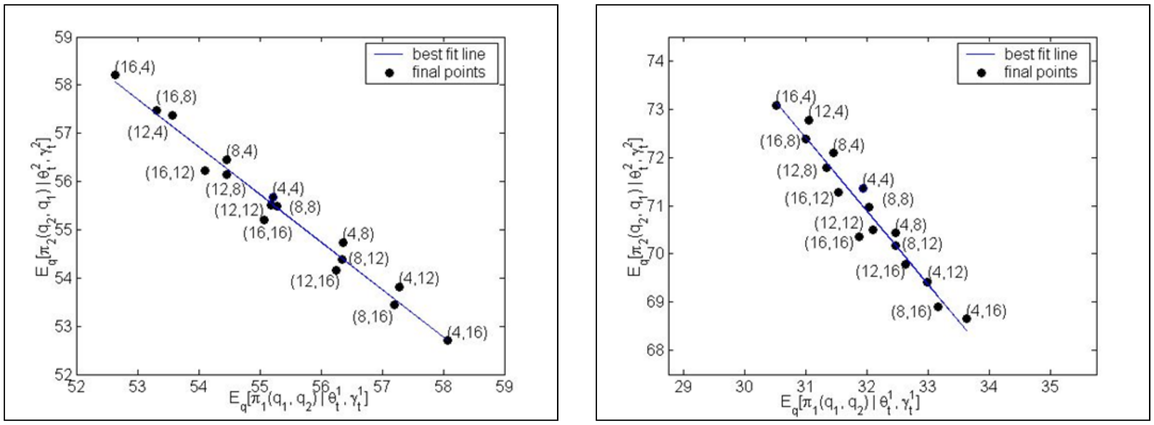
A fourth key feature of the dynamics under Markov Nature regards the sum of the two players' quantities and the sum of their payoffs, and is similar to a feature characteristic of the dynamics under no Nature. As seen in Figure 2.9, in the symmetric cost case, the long-run expected quantities, corresponding to the values at time T=1000, fall along a line with a slope of -0.85 (s.e.=8e-3), while the long-run payoffs fall along a line with slope –0.98 (s.e. = 4e-3). Thus, varying the priors affects the distribution of production and of payoffs between the two firms, but not either a weighted sum of expected quantity produced nor the total payoff achieved. In the asymmetric cost case, the long-run quantities fall along a line with slope –1.38 (s.e. = 2e-4), while the long-run payoffs fall along a line with slope –1.52 (s.e. = 9e-4). Thus, varying the priors affects the distribution of production and of payoffs between the two firms, but not either the weighted sum of expected quantity produced nor the weighted sum of payoff achieved.



(a)



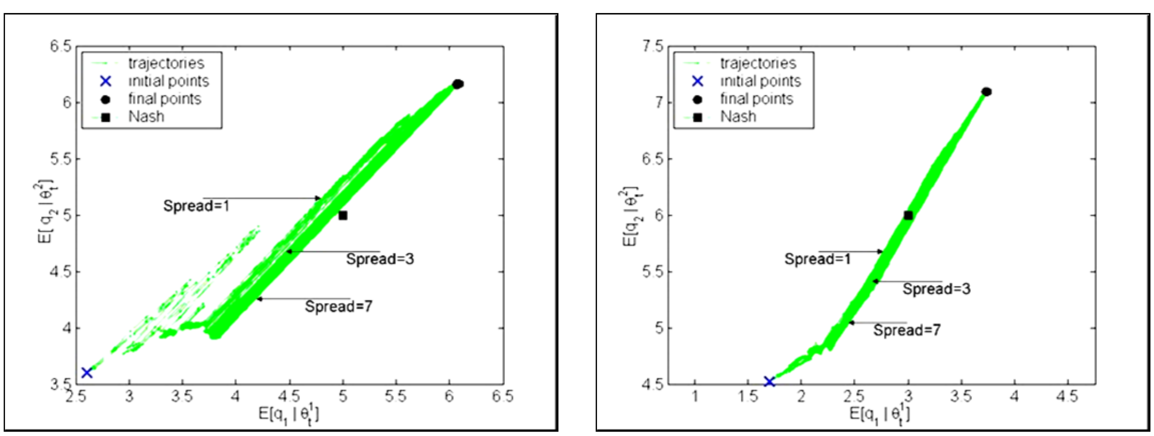
(b)



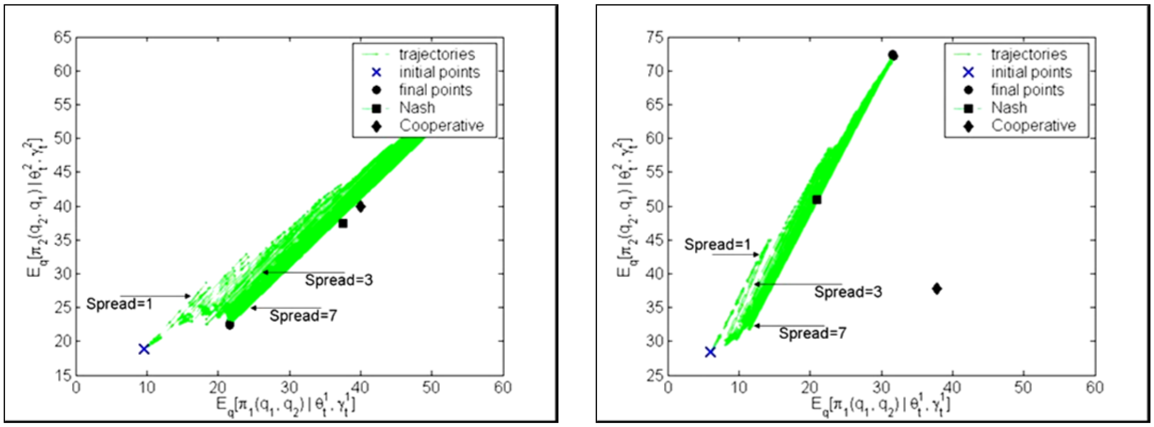
(c) (d)

Figure 2.9. Best fit lines of the endpoints of the trajectories of expected quantity shown in Figure 2.7 when Nature is Markov in the (a) symmetric and (b) asymmetric cases. Plots (c) and (d) give similar best-fit lines for the trajectories of ex ante expected payoff shown in Figure 2.8.

A fifth key feature of the dynamics under Markov Nature is that, as in the no Nature case, varying the spread of each player's prior while holding the mean fixed does not affect the long-run dynamics of play. This result can be gleaned from Figure 2.10.



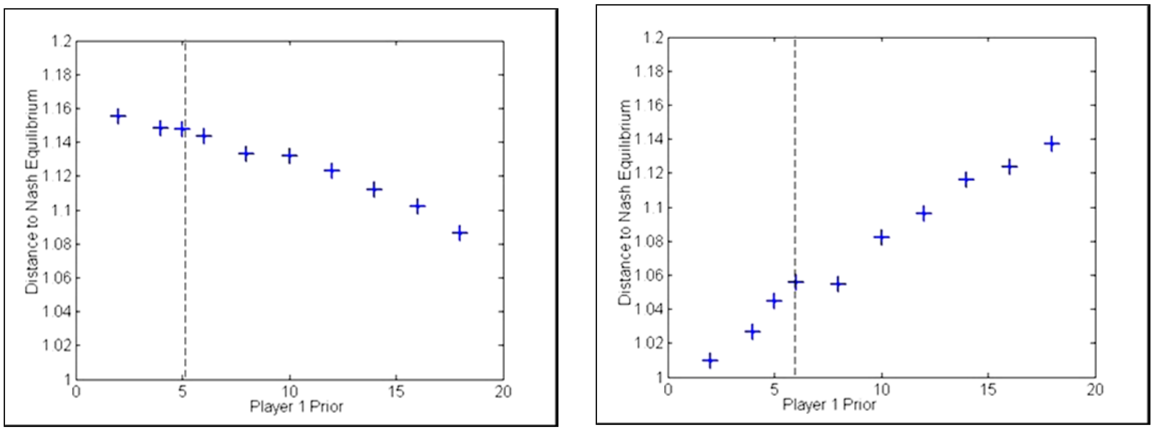
(a) (b)



(c) (d)

Figure 2.10. The effects of varying the spread of player 1's prior around the same mean (q_2 = 10) when Nature is Markov on the trajectories of expected quantity in the (a) symmetric and (b) asymmetric cases, and on the trajectories of ex ante payoff in the (c) symmetric and (d) asymmetric cases.

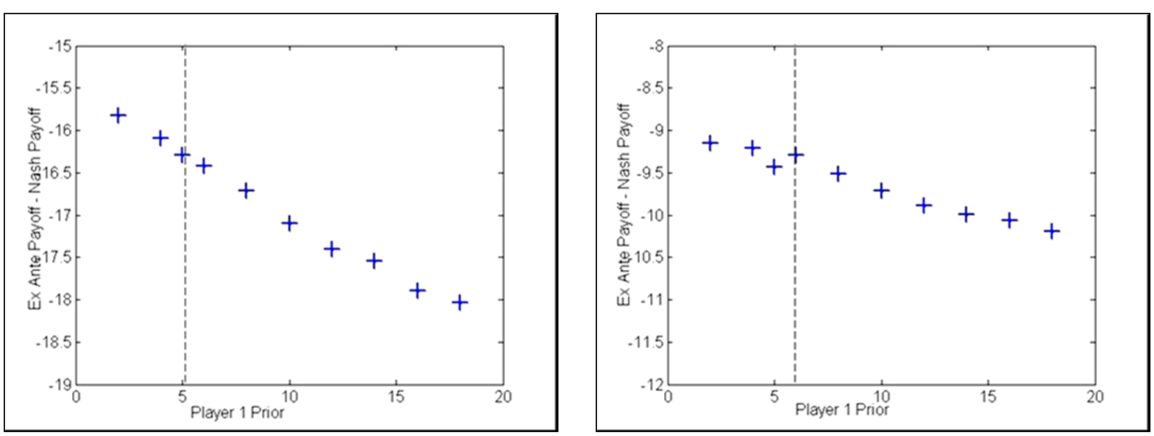
A sixth key feature of the dynamics under Markov Nature is that, unlike in the no Nature case, the distance between player 1's mixed strategy vector and his Nash equilibrium quantity at time T=1000 is no longer inversely related to the difference between the quantity at which player 1's prior is concentrated and player 2's Nash equilibrium quantity. Figure 2.12 plots distance to Nash equilibrium as a function of player 1's (concentrated) prior for a one particular realization of the sequence of shocks under one set of parameters; although the slope of the graph is not robust to the parameters, for both parameter sets the graph is monotonic and therefore does not reach a minimum at player 2's Nash equilibrium quantity.



(a) (b)

Figure 2.12. Distance between player 1's mixed strategy vector and the Nash equilibrium at time T=1000 as a function of player 1's prior when Nature is Markov in the (a) symmetric and (b) asymmetric cases. Player 2's Nash equilibrium strategy is indicated by the dashed line.

A seventh key feature of the dynamics under Markov Nature is that, just as in the no Nature case, there is an *overconfidence premium*: the worse off a player initially expects her opponent to be, the better off she herself will eventually be. As seen in Figure 2.13, the difference between player 1's ex ante payoff and his long-run Nash equilibrium payoff at time T=1000 is decreasing in the quantity at which player 1's prior is concentrated.



(a) (b)

Figure 2.13. Difference between player 1's ex ante payoff and his Nash equilibrium payoff at time T=1000 when Nature is Markov as a function of player 1's prior in the (a) symmetric and (b) asymmetric cases. Player 2's Nash equilibrium strategy is indicated by the dashed line.

In summary, the main results arising from modeling Nature as a Markov process are:

1) Trajectories of play and payoffs are discontinuous and can be viewed as an assemblage of the dynamics that arise when Nature evolves as several separate i.i.d. processes, one for each of the possible values of Nature's previous period play.

2) Neither play nor payoffs converge.

3) Play and payoffs eventually enter respective ergodic sets.

4) Varying the priors affects the distribution of production and of payoffs between the two firms, but not either the weighted sum of expected quantity produced nor the weighted sum of payoff achieved.

5) Varying the spread of each player's prior while holding the mean fixed does not affect the long-run dynamics of play.

6) The distance between player 1's mixed strategy vector and his Nash equilibrium quantity at time T=1000 is *not* inversely related to the difference between the quantity at which player 1's prior is concentrated and player 2's Nash equilibrium quantity.

7) There is an *overconfidence premium*: the worse off a player initially expects her opponent to be, the better off she herself will eventually be.

6. Conclusion

In this paper, I investigate the effects of adding Nature to a stochastic fictitious play model of learning in a static Cournot duopoly game. Nature represents any random state variable that may affect players' payoffs, including global climate change, weather shocks, fluctuations in environmental conditions, natural phenomena, changes to government policy, technological advancement, and macroeconomic conditions.

I develop an agent-based model that enables one to analyze the trajectories and convergence properties of strategies, assessments, and payoffs in logistic smooth fictitious play, and to compare the welfare from logistic smooth fictitious play with that from equilibrium play. I use this agent-based model first to analyze the stochastic fictitious play dynamics in the absence of Nature, and then to investigate the effects of adding Nature.

My analyses yield several central results. First, for both the no Nature and the Markov Nature cases, varying the priors affects the distribution of production and of payoffs between the two firms, but not either the weighted sum of expected quantity produced nor the weighted sum of payoff achieved.

Second, in both the no Nature and the Markov Nature cases, there is an *overconfidence premium*: the worse off a player initially expects her opponent to be, the better off she herself will eventually be. Initial beliefs about the distribution of production and of payoffs can be self-fulfilling.

Third, when Nature is first-order Markov, the trajectories of play and payoffs are discontinuous and can be viewed as an assemblage of the dynamics that arise when Nature evolves as several separate i.i.d. processes, one for each of the possible values of Nature's previous period play. Neither play nor payoffs converge, but they each eventually enter respective ergodic sets.

One key innovation of the work presented in this paper is the agent-based model, which can be used to confirm and visualize existing analytic results, to generate ideas for future analytic proofs, to analyze games for which analytic solutions are difficult to derive, and to aid in the teaching of stochastic fictitious play in a graduate game theory course.

A second innovation of this paper is the analysis of the effects of adding Nature to stochastic fictitious play. In most real-world situations, payoffs are a function not only of the players' strategies and of individual-specific idiosyncratic shocks, but also of common exogenous factors as well. Thus, incorporating Nature brings learning models one step closer to realism.

The results of this paper have important implications for environmental and natural resource issues such as global climate change for which there is uncertainty about the distribution of possible outcomes, and about which agents such as individuals, firms, and/or policy-makers seek to learn.

References

Arieli, I., and Young H.P. (2016). Stochastic learning dynamics and speed of convergence in population games. Econometrica, 84 (2), 627-676.

Doraszelski, U., Lewis, G., and Pakes, A. (2016). Just starting out: Learning and equilibrium in a new market. Working paper, Harvard University.

Erev, I., and Roth, A. (1998). Predicting how people play games: reinforcement learning in experimental games with unique, mixed strategy equilibria. American Economic Review, 88, 848-881.

Fudenberg, D., and Kreps, D. (1993). Learning mixed equilibria. Games and Economic Behavior, 5, 320-367.

Fudenberg, D., and Levine, D. (1999). The theory of learning in games. Cambridge, MA: MIT Press.

Hofbauer, J., and Sandholm, W. (2001). Evolution and learning in games with randomly disturbed payoffs. Working paper. Universität Wien and University of Wisconsin.

Huck, S., Normann, H.-T., and Oeschssler, J. (1999). Learning in Cournot oligopoly—An experiment. Economic Journal, 109 (454), C80-95.

Kalai, E., and Lehrer, E. (1993a). Rational learning leads to Nash equilibrium. Econometrica, 61, 1019-1045.

Kalai, E., and Lehrer, E. (1993b). Subjective equilibrium in repeated games. Econometrica, 61, 1231-1240.

Lee, R.S., and Pakes, A. (2009). Multiple equilibria and selection by learning in an applied setting. Economic Letters, 104 (1), 13-16.

Leoni, P.L. (2008). A constructive proof that learning in repeated games leads to Nash equilibria. The B.E. Journal of Theoretical Economics, 8 (1), Article 29.

Li Calzi, M. (1995). Fictitious play by cases. Games and Economic Behavior, 11 (1), 64-89.

Pakes, A., and McGuire, P. (2001). Stochastic algorithms, symmetric Markov perfect equilibrium, and the “curse” of dimensionality. Econometrica, 69 (5), 1261-1281.

Romaldo, D. (1995). Similarities and evolution. Working paper.

Rudin, W. (1976). Principles of mathematical analysis. New York: McGraw-Hill, Inc.