Appendix: Modeling with the Ordinary Differential Boundary Value Problem approach

Modeling the dynamic oil production decision as a system of ordinary differential equations boundary value problem is based on the work of Harold Hotelling (1939) and Cynthia Lin (2008). We build upon this prior work by developing a model of perfect competition with exogenous price rather than the previous endogenous price models. We also push the limits of complexity in tax policy added to the model structure.

Exogenous price means the transversality condition of price equal to marginal cost, implying shadow price equal to zero, will not work. Additionally, since modeling with a finite horizon imposes an unrealistic a priori end to production, we use an infinite horizon. This essentially amounts to pinning down the front end of the production path with historical first-period production rather than the endpoint with a transversality condition. However, our inability to impose non-negativity constraints on production and reserves remaining in coding the model in Matlab ultimately precluded use of this method.[1]

The exogenous inputs for our modeling are summarized in Table 1. We use the composite cost function, which is comprised of a basic cost function that is scaled by a drilling cost scalar and wells scalar when appropriate, and include royalty and severance taxes. The unit operator's objective function (profit function) is given in equation 26.

1 Objective Function: $\pi(Q_{it}) = P(t)Q_{it}(1 - LR_{it} - ST_{it}ELF_i(Q_i(t)) - CCF(Q_i(t), S_i(t))$

Where Q_{it} is the quantity of production in barrels per month, LR_{it} is the average royalty rate for all leases in the production unit, ST_{it} is the severance tax rate, which is adjusted by the ELF factor that is a function of production rate, and CCF is the composite cost function that is a function of production rate and reserves remaining.

Then the producer's optimal control problem is to choose the production profile {Q(t)} to maximize the present discounted value of the entire stream of profits, given the initial stock S(0) and given the relationship between production Q(t) and the remaining stock S(t), and subject to the constraints that both production and stock are nonnegative. Mathematically, this is written as follows

2 $\text{Max}_{\{S(t)\}} \int_0^\infty [P(t)Q_{it}(1 - LR_{it} - ST_{it}ELF_i(Q_i(t))) - CCF(Q_i(t),S_i(t))]\, e^{-\rho t}\, dt$

s.t. $dS(t)/dt = -Q(t) \quad : p(t)$

$Q(t) \geq 0$

$S(t) \geq 0$

$S(0) = S_o$

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[1] Mathworks, the makers of Matlab, confirmed that non-negativity constraints cannot be imposed with the bvp4c procedure used to solve ordinary differential boundary value problems. Consequently, we were unable to impose constraints to prevent the production path from pushing reserves into negative territory, an illogical result (i.e. producing more oil than is available) that causes negative numbers raised to fractional exponents in the underlying equations (i.e., imaginary numbers).

Where p(t) is a multiplier associated with the equation of motion for the remaining stock S(t). In other words, p(t) is the value of relaxing this constraint; if the quantity of stock remaining in the ground is increased by one unit (i.e., one barrel is not pumped in the current period), then the value of this change is exactly the shadow price of the reserve or p(t).

The three first-order conditions (FOC) for dynamic optimality (from the Maximum Principle) are the following.

1.) Static optimality in the current period implies price equal to marginal cost for perfect competition (i.e., price takers). Alternatively, the shadow price p(t) must equal price minus marginal cost of production.

$$p(t) = P(t) - MC$$

$$p(t) = P(t) - \frac{\partial CCF(S(t),Q(t))}{\partial Q} - P(t)(LR_i + ST_i ELF(Q_i(t))) - ST_i Q_{it} P(t)\frac{\partial ELF}{\partial Q}$$

$$p(t) = P(t)(1 - LR_i - ST_i ELF(Q_i(t))) - ST_i Q_{it} P(t)\frac{\partial ELF}{\partial Q} - \frac{\partial CCF}{\partial Q} \qquad 3$$

Note, the term ($LR_i + ST_i ELF(Q_i(t))$) is an approximation only of "total government take" (a percentage) since we do not include state property tax, state corporate income tax, or federal corporate income tax.

2.) Evolution of the shadow price over time, to ensure inter-period optimality over all finite sub-periods.[2]

$$\frac{dp(t)}{dt} = \frac{\partial CCF(S(t),Q(t))}{\partial S} + \rho p(t) = \frac{\partial CCF}{\partial S} + \rho[P(t)(1 - LR_i - ST_i ELF(Q_i(t))) - \frac{\partial CCF}{\partial Q}] \qquad 4$$

3.) The transversality condition, required for optimality over an infinite time horizon.[3]

$$\lim_{t\to\infty} p(t)S(t)e^{-\rho t} = 0 \qquad 5$$

Rewriting the problem as a boundary value problem with differential equations for Q_{it} (instead of P_{it}) and S_{it} proceeds as follows. Following the methodology from Lin (2008), the Hotelling problem can be reformulated into the following ordinary differential equation boundary value problem.

Step 1: combine FOC 1 and 2 (equations 3 and 4)

$$p(t) = P(t)(1 - LR_i - ST_i ELF(Q_i(t))) - \frac{\partial CCF}{\partial Q} - ST_i Q_{it} P(t)\frac{\partial ELF}{\partial Q} \quad \text{(from FOC 1, equation 3)}$$

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[2] There is a sign change when modeling with reserves remaining (S) rather than cumulative stock extracted (X). Since the change in cumulative stock extracted (X) is the negative of the change in reserves remaining (i.e., a 1-unit increase in X equals a 1-unit decrease in S), then dX/dS = -1. Then dCCF(X)/dS = (dCCF/dX)(dX/dS) = -(dCCF/dX) and dp(t)/dt = dCCF/dS + ρp(t) (without a negative sign on the dC/dS).

[3] The limit goes to zero because the unit operator wants to fully monetize the resource (i.e., the present discounted value of the shadow price times remaining resources is zero).

$$\mathrm{p(t) = P(t) - P(t)LR_i - P(t)ST_iELF(Q(t))} - \frac{\partial C}{\partial Q} - \mathrm{ST_iQ_{it}P(t)}\frac{\partial ELF}{\partial Q}$$

$$\frac{dp(t)}{dt} = \frac{d}{dt}P(t) - \frac{d}{dt}\mathrm{(P(t)LR_i)} - \frac{d}{dt}\mathrm{(P(t)ST_iELF(Q_i(t)))} - \frac{d}{dt}\frac{\partial C}{\partial Q} - \frac{d}{dt}[ST_iQ_i(t)P(t)\frac{\partial ELF}{\partial Q}]$$

$$\frac{dp(t)}{dt} = \frac{d}{dt}P(t) - \frac{d}{dt}\mathrm{(P(t)LR_i)} - \frac{d}{dt}\mathrm{(P(t)ST_iELF(Q_i(t)))} - \frac{d}{dt}\frac{\partial CCF}{\partial Q}$$

$$- [ST_i\frac{\partial Q}{\partial t}P(t)\frac{\partial ELF}{\partial Q} + ST_iQ_i(t)\frac{d}{dt}P(t)\frac{\partial ELF}{\partial Q} + ST_iQ_i(t)P(t)\frac{d}{dt}\frac{\partial ELF}{\partial Q}]$$

6

$$\frac{dp(t)}{dt} = \frac{d}{dt}P(t) - LR_i\frac{d}{dt}P(t) - [ST_iP(t)\frac{d}{dt}(ELF(Q_i(t))) + ELF(Q_i(t))ST_i\frac{d}{dt}P(t)]$$

$$-\frac{d}{dt}\frac{\partial C}{\partial Q} - [ST_i\frac{\partial Q}{\partial t}P(t)\frac{\partial ELF}{\partial Q} + ST_iQ_{it}\frac{d}{dt}P(t)\frac{\partial ELF}{\partial Q} + ST_iQ_{it}P(t)\frac{d}{dt}\frac{\partial ELF}{\partial Q}]$$

The following derivatives are known (see Table 2)

$$\frac{d}{dt}\mathrm{(ELF(Q_i(t)))} = \frac{\partial ELF}{\partial Q}\frac{\partial Q}{\partial t}$$

$$\frac{d}{dt}\frac{\partial CCF}{\partial Q} = \frac{d^2CCF}{dQ^2}\frac{\partial Q}{\partial t} + \frac{d^2CCF}{dSdQ}\frac{\partial S}{\partial t}$$

Then, the previous equation for $\frac{dp(t)}{dt}$ (equation 6) can be re-written as follows.

7

$$\frac{dp(t)}{dt} = \mathrm{(1\text{-}LR_i)}\frac{d}{dt}\mathrm{P(t) - ST_iP(t)}\frac{\partial ELF}{\partial Q}\frac{\partial Q}{\partial t} - \mathrm{ELF(Q_i(t))ST_i}\frac{d}{dt}\mathrm{P(t)} - \frac{d^2C}{dQ^2}\frac{\partial Q}{\partial t} - \frac{d^2C}{dSdQ}\frac{\partial S}{\partial t}$$

$$- \mathrm{ST_iP(t)}\frac{\partial ELF}{\partial Q}\frac{\partial Q}{\partial t} - \mathrm{ST_iQ_i(t)}\frac{d}{dt}\mathrm{P(t)}\frac{\partial ELF}{\partial Q} - \mathrm{ST_iQ_i(t)P(t)}\frac{d}{dt}\frac{\partial ELF}{\partial Q}$$

Now, we substitute equation 7 into the left side of FOC 2 (equation 4)

$$\mathrm{(1\text{-}LR_i\text{-}\ ST_iELF(Q_i(t)))}\frac{d}{dt}\mathrm{P(t) - ST_iP(t)}\frac{\partial ELF}{\partial Q}\frac{\partial Q}{\partial t} - \frac{d^2C}{dQ^2}\frac{\partial Q}{\partial t} - \frac{d^2C}{dSdQ}\frac{\partial S}{\partial t} - \mathrm{ST_i\ P(t)}\frac{\partial ELF}{\partial Q}\frac{\partial Q}{\partial t}$$

$$- \mathrm{ST_iQ_i(t)}\frac{d}{dt}\mathrm{P(t)}\frac{\partial ELF}{\partial Q} - \mathrm{ST_iQ_i(t)P(t)}\frac{d}{dt}\frac{\partial ELF}{\partial Q} = \frac{\partial C}{\partial S} + \rho\mathrm{[P(t)(1\text{-}LR_i\text{-}ST_iELF(Q_i(t))) - }\frac{\partial C}{\partial Q}]$$

However, before using algebra to isolate $\frac{\partial Q}{\partial t}$ on the left side, it is necessary to write out the entire expression for $\frac{d}{dt}\frac{\partial ELF}{\partial Q}$ since it contains $\frac{\partial Q}{\partial t}$ terms (see appendix H). We then expand the expression above in appendix I to derive the following expression for $\frac{\partial Q}{\partial t}$.

8 $$\frac{\partial Q}{\partial t} = \frac{KA - LD + E}{H + IA + GB + FC + JD}$$

Where $E = (1\text{-}LR_i\text{-}ST_iELF(Q_i(t)))\frac{d}{dt}P(t) - ST_iQ_{it}\frac{d}{dt}P(t)\frac{\partial ELF}{\partial Q} - \frac{d^2C}{dSdQ}\frac{\partial S}{\partial t} - \frac{\partial C}{\partial S}$

$- \rho[P(t)(1\text{-}LR_i\text{-}ST_iELF(Q_i(t)))\text{-}\frac{\partial C}{\partial Q}]$

and all other letters in the equation are as given in appendix I, and remain consistent across cost specifications and constant/variable discount rates. Equation 8 is one of the differential equations in our boundary value problem; it contains the information from FOC 1 and 2 (equations 3 and 4).

Step 2: combine FOC 1 and 3 (equations 3 and 5), making sure the limit contains a Q term so the boundary conditions will pin it down.

FOC 1: $p(t) = P(t)(1\text{-}LR_i \text{-}ST_iELF(Q_i(t))) - ST_iQ_{it}P(t)\frac{\partial ELF}{\partial Q} - \frac{\partial C}{\partial Q}$

FOC 3: $\lim_{t\to\infty} p(t)S(t)e^{-\rho t} = 0$

9 $$\lim_{t\to\infty} [P(t)(1 - LR_i - ST_iELF(Q_i(t))) - ST_iQ_{it}P(t)\frac{\partial ELF}{\partial Q} - \frac{\partial C}{\partial Q}]S(t)e^{-\rho t} = 0$$

Equation 9 is one boundary condition in our boundary value problem; it contains the information from FOC 1 and 3, and the term $\frac{\partial C}{\partial Q}$ does contain a Q term.

Step 3: Define the second differential equation and second boundary condition from the maximization constraints.

The second differential equation comes from the fact that the rate of change in reserves remaining (S(t)) is equal to the negative rate of production (Q(t)). Thus, we have,

10 $$\frac{d}{dt}S(t) = -Q(t)$$

The second boundary condition comes directly from the constraints to which the producer's maximization problem is subject. That is,

11 $$S(0) = S_o$$

Finally, the solution to the boundary value problem specified by the two differential equations (equations 8 and 10) and two boundary conditions (equations 9 and 11) is equivalent to the original producer's optimal control problem. The boundary value problem is solved with the software package Matlab with the bvp4c routine. The derivations necessary for this modeling are summarized in Table 2.

Table 1: Exogenous Inputs to Modeling

Functions that are consistent across all fields	
Estimated Functions	**Explanations**
Price$_t$ $= P(t) = c_4YR^2 + c_5YR + c_6$ $c_4 = 0.0517781$ $c_5 = -206.4746$ $c_6 = 205{,}846.7$	Price = wellhead value, 1982-84 $/bbl Yr = year (date, e.g., 1982)
Baseline Total Cost $= BC_i = C(Q_i(t),S_i(t)) = c_1Q_i^{c2}S_i^{c3}$ $c_1 = 178187.2068$ $c_2 = 1.000529$ $c_3 = -0.548916$	Total facilities cost of production Total cost ($, 1982-84), ($/bbl)*Q
API Drilling Cost Scalar $DCS = c_7 + c_8YrIL + c_9YrIL^2 + c_{10}YrIL^3 + c_{11}YrIL^4 + c_{12}YrIL^5 + c_{13}YrIL^6$ $c_7 = 1.413501$ $c_8 = -0.5839932$ $c_9 = 0.161024$ $c_{10} = -0.0175783$ $c_{11} = 0.0008877$ $c_{12} = -0.0000211$ $c_{13} = 0.000000192$	Applied only from 1969 to 2004 Indexed Year = YRI = (Year – 1968) Optional lag: Lag in years (e.g., 1) Indexed & Lagged Year = YRIL = (Year – 1968 – lag)
Dampened Drilling Cost Scalar $DDCS = 1+(DCS-1)/Dmp$	The Dampener (Dmp) reduces the magnitude of scalar deviations around 1 (e.g., dampener = 2 cuts magnitude in half).

Field-Specific Elements

Estimated Functions

Historical Wells $= HW = c_{14} + c_{15}*QI + c_{16}*SI$

	c_{14}	c_{15}	c_{16}
Badami (BHW):	$c_{14} = 4.235744$	$c_{15} = 8.210379$	$c_{16} = 51.36989$
Colville (CHW):	$c_{14} = 83.06196$	$c_{15} = 1.718612$	$c_{16} = -151.6971$
Endicott (EHW):	$c_{14} = 62.86919$	$c_{15} = 9.693985$	$c_{16} = -107.5757$
Kuparuk (KHW):	$c_{14} = 1882.539$	$c_{15} = 18.81515$	$c_{16} = -176.0778$
Milne (MHW):	$c_{14} = 238.7935$	$c_{15} = 49.09711$	$c_{16} = -277.0513$
Northstar (NHW):	$c_{14} = 16.87012$	$c_{15} = 2.481977$	$c_{16} = -125.87012$
Prudhoe (PHW):	$c_{14} = 1087.458$	$c_{15} = 5.359837$	$c_{16} = -78.11036$

Explanations

"Historical Wells" functions define flat planes in the Q,S,Wells space (i.e., constant returns to scale)

Q Index = QI = Q/1,000,000
= million bbl/mo.
S Index = SI = S/1,000,000,000
= billlion bbl res. Rem.

Wells (W)

BadamiWells = BW

$= c_{17} + c_{18*}QI + c_{19}*SI^2 + c_{20}*QI^2*SI^2 + c_{21}*QI^3$

$c_{17} = 2.823319$ $c_{18} = 48.87226$ $c_{19} = 5693.26$ $c_{20} = -2447862$ $c_{21} = 1616.327$

ColvilleWells = CW

$= c_{17} + c_{18}QI + c_{19}QI^2 + c_{20}QI^3 + c_{21}SI + c_{22}SI^2$

$c_{17} = 70.13352$ $c_{18} = 5.993816$ $c_{19} = -1.729683$ $c_{20} = 0.2007895$

$c_{21} = -83.5993$ $c_{22} = -105.38$

EndicottWells = EW

$= c_{17}+c_{18}QI*SI+c_{19}QI*SI^2+c_{20}QI+c_{21}QI^2+c_{22}QI^3+c_{23}SI$

$c_{17} = 66.16589$ $c_{18} = 179.3657$ $c_{19} = -214.7895$ $c_{20} = -7.839296$

$c_{21} = -9.65297$ $c_{22} = 1.21689$ $c_{23} = -114.1997$

KuparukWells = KW

$= c_{17} + c_{18}QI + c_{19}QI^2 + c_{20}QI^3 + c_{21}SI$

$c_{17} = 1345.958$ $c_{18} = 139.626$ $c_{19} = -12.91675$ $c_{20} = 0.3473245$ $c_{21} = -151.1711$

MilneWells = MW

$= c_{17} + c_{18}QI*SI + c_{19}QI^2*SI + c_{20}QI^3$

$c_{17} = -5.005079$ $c_{18} = 222.55$ $c_{19} = -210.2137$ $c_{20} = 62.8936$

NorthstarWells = NW

$= c_{17} + c_{18}QI + c_{19}QI^2 + c_{20}QI^3 + c_{21}QI*SI$

$c_{17} = 0.8447826$ $c_{18} = 21.46647$ $c_{19} = -7.329782$ $c_{20} = 1.265286$ $c_{21} = -81.79707$

PrudhoeWells = PW

$= c_{17} + c_{18}Q + c_{19}Q^2 + c_{20}Q^3 + c_{21}S + c_{22}S^2 + c_{23}S^3$

c_{17} = 31.68469 c_{18} = 0.0000143 c_{19} = -4.99*10^(-13) c_{20} = 5.75*10^(-21)

c_{21} = 3.69*10^(-7) c_{22} = -5.14*10^(-17) c_{23} = 1.78*10^(-27)

“Wells” functions define surfaces in the Q, S, Wells space with decreasing returns to scale.

Note the Prudhoe Wells function was estimated with Q and S rather than the indexed variables QI and SI.

Table 2: Derivations

Derivations that are consistent across all fields

Derivations

Wells Scalar (WS)

WS = 1 if (HW*DRTS_M) > W
WS = W/(HW*DRTS_M) otherwise

Badami = BWS, Colville = CWS, Endicott = EWS, Kuparuk = KWS, Milne = MWS, Northstar = NWS, Prudhoe = PWS

Composite Cost Function (CCF): Variant 1: CCF = BC

- The year is after 2004, so the drilling cost scalar is not used
- The HW*DRTS_M > W, so the WS function = 1 (omitted)

$$\frac{\partial CCF(Q(t),S(t))}{\partial Q} = \frac{\partial CCF}{\partial Q} = c_1 c_2 Q_i^{c_2-1} S_i^{c_3}$$

$$\frac{\partial CCF(Q(t),S(t))}{\partial S} = \frac{\partial CCF}{\partial S} = c_1 c_3 Q_i^{c_2} S_i^{c_3-1}$$

$$\frac{d}{dt}\frac{\partial CCF}{\partial Q} = \frac{\partial \frac{\partial CCF}{\partial Q}}{\partial Q}\frac{\partial Q}{\partial t} + \frac{\partial \frac{\partial CCF}{\partial Q}}{\partial S}\frac{\partial S}{\partial t} = \frac{d^2 CCF}{dQ^2}\frac{\partial Q}{\partial t} + \frac{d^2 CCF}{dSdQ}\frac{\partial S}{\partial t}$$

$$\frac{d^2 CCF}{dQ^2} = (c_2 - 1) c_1 c_2 Q_i^{c_2-2} S_i^{c_3} = \frac{(c_2-1)\frac{\partial CCF}{\partial Q}}{Q}$$

$$\frac{d^2 CCF}{dSdQ} = c_1 c_2 c_3 Q_i^{(c_2-1)} S_i^{(c_3-1)} = \frac{c_2 \frac{\partial CCF}{\partial S}}{Q}$$

Notes and Explanations

This definition becomes field-specific since the HW and W functions are field-specific.

DRTS_M shifts the historical wells plan up or down, changing the margin above (or below) the historical number of wells at which decreasing returns to scale set in (i.e., when the W function exceeds the HW function).

The following Derivatives remain true for all four cost function variations:

$$\frac{dS}{dt} = -Q\,,\ \frac{d}{dt}P(t) = 2c_4 YR + c_5$$

Composite Cost Function (CCF): Variant 2: CCF = BC * DDCS

- The year is pre-2005, so the drilling cost scalar <u>is</u> used
- The HW*DRTS_M > W, so the WS function = 1 (omitted)

$$\frac{\partial CCF}{\partial Q} = \frac{\partial BC}{\partial Q} DDCS + BC * \frac{\partial DDCS}{\partial Q} = \frac{\partial BC}{\partial Q} DDCS$$

$$\frac{\partial CCF}{\partial S} = \frac{\partial BC}{\partial S} DDCS + BC * \frac{\partial DDCS}{\partial S} = \frac{\partial BC}{\partial S} DDCS$$

$$\frac{d}{dt}\frac{\partial CCF}{\partial Q} = \frac{d^2 CCF}{dQ^2}\frac{\partial Q}{\partial t} + \frac{d^2 CCF}{dSdQ}\frac{\partial S}{\partial t} + \frac{\partial^2 CCF}{\partial YrIL \partial Q}\frac{\partial YrIL}{\partial t}$$

$$\frac{d^2 CCF}{dQ^2} = \frac{\partial}{\partial Q}\left(\frac{\partial BC}{\partial Q} DDCS\right) = \frac{\partial^2 BC}{\partial Q^2} DDCS + \frac{\partial BC}{\partial Q}\frac{\partial DDCS}{\partial Q} = \frac{\partial^2 BC}{\partial Q^2} DDCS$$

$$\frac{d^2 CCF}{dSdQ} = \frac{\partial}{\partial S}\left(\frac{\partial BC}{\partial Q} DDCS\right) = \frac{\partial^2 BC}{\partial S \partial Q} DDCS + \frac{\partial BC}{\partial Q}\frac{\partial DDCS}{\partial S} = \frac{\partial^2 BC}{\partial S \partial Q} DDCS$$

$$\frac{d^2 CCF}{dYrILdQ} = \frac{\partial^2 BC}{\partial YrIL \partial Q} DDCS + \frac{\partial DDCS}{\partial YrIL}\frac{\partial BC}{\partial Q} = \frac{\partial DDCS}{\partial YrIL}\frac{\partial BC}{\partial Q}$$

$$\frac{\partial DDCS}{\partial YrIL} = \left(\frac{1}{Dmp}\right)(c_8 + 2c_9 YrIL + 3c_{10} YrIL^2 + 4c_{11} YrIL^3 + 5c_{12} YrIL^4 + 6c_{13} YrIL^5)$$

$$\frac{\partial YrIL}{\partial t} = 0.08333$$

$$\frac{\partial DDCS}{\partial Q} = 0$$

$$\frac{\partial DDCS}{\partial S} = 0$$

The term $\frac{\partial^2 CCF}{\partial YrIL \partial Q}\frac{\partial YrIL}{\partial t}$ is included because the DDCS function has the variable YrIL.

$$\frac{\partial DDCS}{\partial Q} = 0, \quad \frac{\partial DDCS}{\partial S} = 0$$

$$\frac{\partial^2 BC}{\partial YrIL \partial Q} = 0$$

YrIL is measured in years and *t* is measured in months, so the change in year for a change in month is (1/12) = 0.08333.

Composite Cost Function (CCF): Variant 3: CCF = BC * WS

- The year is after 2004, so the drilling cost scalar is not used
- The HW*DRTS_M < W, so the WS function <u>is</u> used

$$\frac{\partial CCF}{\partial Q} = \frac{\partial BC}{\partial Q} WS + BC * \frac{\partial WS}{\partial Q}$$

$$\frac{\partial CCF}{\partial S} = \frac{\partial BC}{\partial S} WS + BC * \frac{\partial WS}{\partial S}$$

$$\frac{d}{dt}\frac{\partial CCF}{\partial Q} = \frac{d^2 CCF}{dQ^2}\frac{\partial Q}{\partial t} + \frac{d^2 CCF}{dSdQ}\frac{\partial S}{\partial t}\ \frac{d^2 CCF}{dQ^2} = \frac{\partial}{\partial Q}\left(\frac{\partial BC}{\partial Q} DCS + BC\frac{\partial WS}{\partial Q}\right)$$

$$= \frac{\partial^2 BC}{\partial Q^2} WS + \frac{\partial BC}{\partial Q}\frac{\partial WS}{\partial Q} + \frac{\partial BC}{\partial Q}\frac{\partial WS}{\partial Q} + BC\frac{\partial^2 WS}{\partial Q^2}$$

$$= \frac{\partial^2 BC}{\partial Q^2} WS + 2\frac{\partial BC}{\partial Q}\frac{\partial WS}{\partial Q} + BC\frac{\partial^2 WS}{\partial Q^2}$$

$$\frac{d^2 CCF}{dSdQ} = \frac{\partial}{\partial S}\left(\frac{\partial BC}{\partial Q} WS + BC\frac{\partial WS}{\partial Q}\right)$$

$$= \frac{\partial^2 BC}{\partial S \partial Q} WS + \frac{\partial BC}{\partial Q}\frac{\partial WS}{\partial S} + \frac{\partial BC}{\partial S}\frac{\partial WS}{\partial Q} + BC\frac{\partial^2 WS}{\partial S \partial Q}$$

Composite Cost Function (CCF), Variant 4: CCF=BC*DDCS*WS

- The year is pre-2005, so the drilling cost scalar <u>is</u> used
- The HW*DRTS_M < W, so the WS function <u>is</u> used

$$\frac{\partial CCF}{\partial Q} = \frac{\partial BC}{\partial Q}\cdot DDCS \cdot WS + BC \cdot \frac{\partial DDCS}{\partial Q}\cdot WS + BC \cdot DDCS \cdot \frac{\partial WS}{\partial Q}$$

$$= \frac{\partial BC}{\partial Q}\cdot DDCS \cdot WS + BC \cdot DDCS \cdot \frac{\partial WS}{\partial Q}$$

Where,

$WS, \frac{\partial WS}{\partial Q}, \frac{\partial WS}{\partial S}, \frac{\partial^2 WS}{\partial Q^2}, and\ \frac{\partial^2 WS}{\partial S \partial Q}$

are field-specific

$$\frac{\partial CCF}{\partial S} = \frac{\partial BC}{\partial S}\cdot DDCS\cdot WS + BC\cdot\frac{\partial DDCS}{\partial S}\cdot WS + BC\cdot DDCS\cdot\frac{\partial WS}{\partial S}$$

$$= \frac{\partial BC}{\partial S}\cdot DDCS\cdot WS + BC\cdot DDCS\cdot\frac{\partial WS}{\partial S}$$

$$\frac{d}{dt}\frac{\partial CCF}{\partial Q} = \frac{d^2 CCF}{dQ^2}\frac{\partial Q}{\partial t} + \frac{d^2 CCF}{dSdQ}\frac{\partial S}{\partial t} + \frac{\partial^2 CCF}{\partial YrIL\partial Q}\frac{\partial YrIL}{\partial t}$$

$$\frac{d^2 CCF}{dQ^2} = \frac{\partial}{\partial Q}\left(\frac{\partial BC}{\partial Q}\cdot DDCS\cdot WS + BC\cdot DDCS\cdot\frac{\partial WS}{\partial Q}\right)$$

$$= \frac{\partial^2 BC}{\partial Q^2}\cdot DDCS\cdot WS + \frac{\partial BC}{\partial Q}\frac{\partial DDCS}{\partial Q}\cdot WS + \frac{\partial BC}{\partial Q}\cdot DDCS\cdot\frac{\partial WS}{\partial Q}$$

$$+ \frac{\partial BC}{\partial Q}\cdot DDCS\cdot\frac{\partial WS}{\partial Q} + BC\cdot\frac{\partial DDCS}{\partial Q}\frac{\partial WS}{\partial Q} + BC\cdot DDCS\cdot\frac{\partial^2 WS}{\partial Q^2}$$

$$= \frac{\partial^2 BC}{\partial Q^2}\cdot DDCS\cdot WS + 2\frac{\partial BC}{\partial Q}\cdot DDCS\cdot\frac{\partial WS}{\partial Q} + BC\cdot DDCS\cdot\frac{\partial^2 WS}{\partial Q^2}$$

$$\frac{d^2 CCF}{dSdQ} = \frac{\partial}{\partial S}\left(\frac{\partial BC}{\partial Q}\cdot DDCS\cdot WS + BC\cdot DDCS\cdot\frac{\partial WS}{\partial Q}\right)$$

$$= \frac{\partial^2 BC}{\partial S\partial Q}\cdot DDCS\cdot WS + \frac{\partial BC}{\partial Q}\frac{\partial DDCS}{\partial S}\cdot WS + \frac{\partial BC}{\partial Q}\cdot DDCS\cdot\frac{\partial WS}{\partial S}$$

$$+ \frac{\partial BC}{\partial S}\cdot DDCS\cdot\frac{\partial WS}{\partial Q} + BC\cdot\frac{\partial DDCS}{\partial S}\frac{\partial WS}{\partial Q} + BC\cdot DDCS\cdot\frac{\partial^2 WS}{\partial S\partial Q}$$

$$= \frac{\partial^2 BC}{\partial S\partial Q} DDCS\cdot WS + \frac{\partial BC}{\partial Q}\frac{\partial WS}{\partial S} DDCS + \frac{\partial BC}{\partial S}\frac{\partial WS}{\partial Q} DDCS$$

$$+ \frac{\partial^2 WS}{\partial S\partial Q} BC\cdot DDCS$$

$$\frac{\partial DDCS}{\partial Q} = 0$$

$$\frac{\partial DDCS}{\partial S} = 0$$

The term $\frac{\partial^2 CCF}{\partial YrIL\partial Q}\frac{\partial YrIL}{\partial t}$ is included because the DDCS function has the variable YrIL.

$$\frac{\partial DDCS}{\partial Q} = 0$$

$$\frac{\partial DDCS}{\partial S} = 0$$

$$\frac{d^2CCF}{dYrILdQ} = \frac{\partial BC}{\partial Q}\frac{\partial DDCS}{\partial YrIL}WS + BC\frac{\partial DDCS}{\partial YrIL}\frac{\partial WS}{\partial Q}$$

$$\frac{\partial DDCS}{\partial YrIL} = (\frac{1}{Dmp})(c_8 + 2c_9YrIL + 3c_{10}YrIL^2 + 4c_{11}YrIL^3 + 5c_{12}YrIL^4 + 6c_{13}YrIL^5)$$

$$\frac{\partial YrIL}{\partial t} = 0.08333$$

where $WS, \frac{\partial WS}{\partial Q}, \frac{\partial WS}{\partial S}, \frac{\partial^2 WS}{\partial Q^2}, and\ \frac{\partial^2 WS}{\partial S\partial Q}$ are field-specific

$$\frac{\partial^2 BC}{\partial YrIL\partial Q} = 0,\ \frac{\partial WS}{\partial YrIL} = 0,\ \frac{\partial BC}{\partial YrIL} = 0,\ \frac{\partial^2 WS}{\partial YrIL\partial Q} = 0$$

YrIL is measured in years and *t* is measured in months, so the change in year for a change in month is (1/12) = 0.08333

For all fields, the following functional forms are consistent

$HW = c_{14} + c_{15}QI + c_{16}SI = c_{14} + (c_{15}/10^6)Q + (c_{16}/10^9)S$ ➔ $\frac{\partial HW}{\partial Q} = \frac{c_{15}}{10^6}$ and $\frac{\partial HW}{\partial S} = \frac{c_{16}}{10^9}$ and $\frac{\partial^2 HW}{\partial Q^2} = 0$ and $\frac{\partial^2 HW}{\partial S\partial Q} = 0$

WS = 1 if (HW*DRTS_M) > W
= W/(HW*DRTS_M) otherwise

Thus, if WS = 1, $\frac{\partial WS}{\partial Q} = \frac{\partial WS}{\partial S} = \frac{\partial^2 WS}{\partial Q^2} = \frac{\partial^2 WS}{\partial S\partial Q} = 0$ and Otherwise, $\frac{\partial WS}{\partial Q} = (\frac{1}{DRTS\_M})[\frac{\partial W}{\partial Q}HW^{-1} - \frac{\partial HW}{\partial Q}\cdot W\cdot HW^{-2}]$

$$\frac{\partial WS}{\partial S} = (\frac{1}{DRTS\_M})[\frac{\partial W}{\partial S}HW^{-1} - \frac{\partial HW}{\partial S}\cdot W\cdot HW^{-2}]$$

$$\frac{\partial^2 WS}{\partial Q^2} = (\frac{1}{DRTS\_M})[\frac{\partial^2 W}{\partial Q^2}HW^{-1} - 2\frac{\partial W}{\partial Q}\frac{\partial HW}{\partial Q}HW^{-2} + 2\cdot W\cdot HW^{-3}(\frac{\partial HW}{\partial Q})^2]$$

$$\frac{\partial^2 WS}{\partial S\partial Q} = (\frac{1}{DRTS\_M})[\frac{\partial^2 W}{\partial S\partial Q}HW^{-1} - \frac{\partial W}{\partial Q}\frac{\partial HW}{\partial S}HW^{-2} - \frac{\partial W}{\partial S}\frac{\partial HW}{\partial Q}HW^{-2} + 2\cdot W\cdot HW^{-3}\frac{\partial HW}{\partial Q}\frac{\partial HW}{\partial S}]$$

The economic limit factor (ELF) formula in Alaska Statute (AS 43.55.150) presented unique challenges for finding derivatives.

$$\mathrm{ELF}(Q_i(t)) = \left(1-\frac{300*WELLS_{it}*DAYS_t}{Q_{it}}\right)^{\left(\frac{150{,}000}{Q_{it}/DAYS_t}\right)^{\left(\frac{460*WELLS_{it}*DAYS_t}{300*WELLS_{it}*DAYS_t}\right)}} = \left(1-\frac{300*WELLS_{it}*DAYS_t}{Q_{it}}\right)^{\left(\frac{150{,}000}{Q_{it}/DAYS_t}\right)^{1.5333}}$$

$$\frac{\partial ELF(Q_i(t))}{\partial Q} = \frac{\partial ELF}{\partial Q} = [[(vu^{v-1})(\frac{du}{dQ}) + (\ln u)(u^v)(\frac{dv}{dQ})][\frac{\partial}{\partial Q}g(f(Q))][\frac{\partial}{\partial Q}f(Q)] \qquad \text{(See Appendix H)}$$

$$\frac{d}{dt}\frac{\partial ELF}{\partial Q} = \{\frac{d}{dt}[(vu^{v-1})(\frac{du}{dQ})] + \frac{d}{dt}[(\ln u)(u^v)(\frac{dv}{dQ})]\}[\frac{\partial f(Q)}{\partial Q}] + \{\frac{d}{dt}[\frac{\partial f(Q)}{\partial Q}]\}[(vu^{v-1})(\frac{du}{dQ}) + (\ln u)(u^v)(\frac{dv}{dQ})] \quad \text{(See Appendix H)}$$

Supporting Field-Specific Derivations

Derivations

$BW = c_{17} + c_{18}*QI + c_{19}*SI^2 + c_{20}*QI^2*SI^2 + c_{21}*QI^3 = c_{17} + (c_{18}/10^6)Q + (c_{19}/10^{18})S^2 + (c_{20}/10^{30})Q^2S^2 + (c_{21}/10^{18})Q^3$

$$\frac{\partial BW}{\partial Q} = \frac{c_{18}}{10^6} + (\frac{2c_{20}}{10^{30}})QS^2 + (\frac{3c_{21}}{10^{18}})Q^2 \qquad \frac{\partial BW}{\partial S} = (\frac{2c_{19}}{10^{18}})S + (\frac{2c_{20}}{10^{30}})Q^2S$$

$$\frac{\partial^2 BW}{\partial Q^2} = (\frac{2c_{20}}{10^{30}})S^2 + (\frac{6c_{21}}{10^{18}})Q \qquad \frac{\partial^2 BW}{\partial S\partial Q} = (\frac{4c_{20}}{10^{30}})QS$$

$CW = c_{17} + c_{18}QI + c_{19}QI^2 + c_{20}QI^3 + c_{21}SI + c_{22}SI^2 = c_{17}+(c_{18}/10^6)Q+(c_{19}/10^{12})Q^2+(c_{20}/10^{18})Q^3+(c_{21}/10^9)S+(c_{22}/10^{18})S^2$

$$\frac{\partial CW}{\partial Q} = \frac{c_{18}}{10^6} + (\frac{2c_{19}}{10^{12}})Q + (\frac{3c_{20}}{10^{18}})Q^2 \qquad \frac{\partial CW}{\partial S} = (\frac{c_{21}}{10^9}) + (\frac{2c_{22}}{10^{18}})S$$

$$\frac{\partial^2 CW}{\partial Q^2} = (\frac{2c_{19}}{10^{12}}) + (\frac{6c_{20}}{10^{18}})Q \qquad \frac{\partial^2 CW}{\partial S\partial Q} = 0$$

$EW = c_{17}+c_{18}QI*SI+c_{19}QI*SI^2+c_{20}QI+c_{21}QI^2+c_{22}QI^3+c_{23}SI = c_{17} +(c_{18}/10^{15})QS+(c_{19}/10^{24})QS^2 +(c_{20}/10^6)Q+(c_{21}/10^{12})Q^2+(c_{22}/10^{18})Q^3 +(c_{23}/10^9)S$

$$\frac{\partial EW}{\partial Q} = (\frac{c_{18}}{10^{15}})S + (\frac{c_{19}}{10^{24}})S^2 + (\frac{c_{20}}{10^6}) + (\frac{2c_{21}}{10^{12}})Q + (\frac{3c_{22}}{10^{18}})Q^2 \qquad \frac{\partial EW}{\partial S} = (\frac{c_{18}}{10^{15}})Q + (\frac{2c_{19}}{10^{24}})QS + (\frac{c_{23}}{10^9})$$

$$\frac{\partial^2 EW}{\partial Q^2} = (\frac{2c_{21}}{10^{12}}) + (\frac{6c_{22}}{10^{18}})Q \qquad \frac{\partial^2 EW}{\partial S \partial Q} = (\frac{c_{18}}{10^{15}}) + (\frac{2c_{19}}{10^{24}})S$$

$KW = c_{17} + c_{18}QI + c_{19}QI^2 + c_{20}QI^3 + c_{21}SI = c_{17} + (c_{18}/10^6)Q + (c_{19}/10^{12})Q^2 + (c_{20}/10^{18})Q^3 + (c_{21}/10^9)S$

$$\frac{\partial KW}{\partial Q} = (\frac{c_{18}}{10^6}) + (\frac{2c_{19}}{10^{12}})Q + (\frac{3c_{20}}{10^{18}})Q^2 \qquad \frac{\partial KW}{\partial S} = (\frac{c_{21}}{10^9})$$

$$\frac{\partial^2 KW}{\partial Q^2} = (\frac{2c_{19}}{10^{12}}) + (\frac{6c_{20}}{10^{18}})Q \qquad \frac{\partial^2 KW}{\partial S \partial Q} = 0$$

$MW = c_{17} + c_{18}QI*SI + c_{19}QI^2*SI + c_{20}QI^3 = c_{17} + (c_{18}/10^{15})Q*S + (c_{19}/10^{21})Q^2*S + (c_{20}/10^{18})Q^3$

$$\frac{\partial MW}{\partial Q} = (\frac{c_{18}}{10^{15}})S + (\frac{2c_{19}}{10^{21}})QS + (\frac{3c_{20}}{10^{18}})Q^2 \qquad \frac{\partial MW}{\partial S} = (\frac{c_{18}}{10^{15}})Q + (\frac{c_{19}}{10^{21}})Q^2$$

$$\frac{\partial^2 MW}{\partial Q^2} = (\frac{2c_{19}}{10^{21}})S + (\frac{6c_{20}}{10^{18}})Q \qquad \frac{\partial^2 MW}{\partial S \partial Q} = (\frac{c_{18}}{10^{15}}) + (\frac{2c_{19}}{10^{21}})Q$$

$NW = c_{17} + c_{18}QI + c_{19}QI^2 + c_{20}QI^3 + c_{21}QI*SI = c_{17} + (c_{18}/10^6)Q + (c_{19}/10^{12})Q^2 + (c_{20}/10^{18})Q^3 + (c_{21}/10^{15})Q*S$

$$\frac{\partial NW}{\partial Q} = (\frac{c_{18}}{10^6}) + (\frac{2c_{19}}{10^{12}})Q + (\frac{3c_{20}}{10^{18}})Q^2 + (\frac{c_{21}}{10^{15}})S \qquad \frac{\partial NW}{\partial S} = (\frac{c_{21}}{10^{15}})Q$$

$$\frac{\partial^2 NW}{\partial Q^2} = (\frac{2c_{19}}{10^{12}}) + (\frac{6c_{20}}{10^{18}})Q \qquad \frac{\partial^2 NW}{\partial S \partial Q} = (\frac{c_{21}}{10^{15}})$$

$PW = c_{17} + c_{18}Q + c_{19}Q^2 + c_{20}Q^3 + c_{21}S + c_{22}S^2 + c_{23}S^3$

$$\frac{\partial PW}{\partial Q} = c_{18} + 2c_{19}Q + 3c_{20}Q^2 \qquad \frac{\partial PW}{\partial S} = c_{21} + 2c_{22}S + 3c_{23}S^2$$

$$\frac{\partial^2 PW}{\partial Q^2} = 2c_{19} + 6c_{20}Q \qquad \frac{\partial^2 NW}{\partial S \partial Q} = 0$$

Derivations of ELF

In the ELF function, WELLS is a function of Q and S or X. Thus, we have

$$\text{ELF(Q,X)} = (1 - \frac{300 * WELLS(Q,X) * DAYS_t}{Q_{it}})^{(\frac{150,000}{Q_{it}/DAYS_t})^{1.5333}}$$

Estimate WELLS(Q,X) as WELLS = $c_7Q^{c8}X^{c9}$

- nonlinear relationship between wells and Q since average well size (bbl/day) probably increases with Q since production is likely coming from a larger and/or more productive field where each well can produce more.
- Nonlinear relationship between wells and X since the need for additional wells to maintain production at some level likely increases nonlinearly as the field is depleted.
- Estimate with linear regression: Log(WELLS) = log(c_7) + $c_8$log(Q) + $c_9$log(X)

Then $\frac{\partial WELLS}{\partial Q} = c_7c_8Q^{c8-1}X^{c9}$

Let f(Q) = $\frac{WELLS}{Q} = (c_7Q^{c8}X^{c9})/Q$

g(f(Q) = (1 – 300*DAYS*f(Q))

h(g(f(Q)) = $[g(f(Q))]^{(150,000*DAYS*Q^{-1})^{1.5333}}$

Chain Rule:

F = g(f(Q)) F’ = g’(f(Q))f’(Q)

and

G = h(g(f(Q))) = ELF G’ = h’(g(f(Q)))[g(f(Q))]’

then

G’ = $\frac{\partial ELF}{\partial Q}$ = h’(g(f(Q)))g’(f(Q))f’(Q)

Then,

f’(Q) = $c_7(c_8-1)Q^{c8-2}X^{c9}$

g’(f(Q)) = (1 – 300*DAYS*f(Q))’ = 300*DAYS

$$\text{h'(g(f(Q)))} = \frac{\partial}{\partial \text{Q}}[\text{g(f(Q))}]^{(150{,}000*\text{DAYS}*\text{Q}^{-1})^{1.5333}}$$

if u, v are both functions of Q, then

$$\frac{\partial}{\partial \text{Q}}(\text{u}^{\text{v}}) = vu^{v-1}\frac{du}{dQ} + (\ln u)(u^{v})(\frac{dv}{dQ})$$

Let u = g(f(Q))

Let v = $(150{,}000*\text{Days}*\text{Q}^{-1})^{1.5333}$

Then $\frac{\partial u}{\partial Q} = 300 * DAYS$

And $\frac{\partial v}{\partial Q} = 1.5333(150{,}000 * DAYS * Q^{-1})^{0.5333}(-150{,}000 * DAYS * Q^{-2})$

Then h'(g(f(Q))) = $\frac{\partial}{\partial Q}(u^{v}) = vu^{v-1}\frac{du}{dQ} + (\ln u)(u^{v})(\frac{dv}{dQ})$

Thus, $\frac{\partial ELF}{\partial Q} = [[(vu^{v-1})(\frac{du}{dQ}) + (\ln u)(u^{v})(\frac{dv}{dQ})][\frac{\partial}{\partial Q}g(f(Q))][\frac{\partial}{\partial Q}f(Q)]$

Where $\text{f(Q)} = \frac{WELLS}{Q} = (c_7Q^{c8}X^{c9})/Q = c_7Q^{c8-1}X^{c9}$

$$\frac{\partial}{\partial Q}f(Q) = c_7(c_8 - 1)Q^{c_8-2}X^{c_9}$$

g(f(Q)) = (1 – 300*DAYS*f(Q))

$$\frac{\partial}{\partial Q}g(f(Q)) = 300 * DAYS$$

u = g(f(Q)) = (1 – 300*DAYS*f(Q))

v = $(150{,}000*\text{Days}*\text{Q}^{-1})^{1.5333}$

$$\frac{\partial u}{\partial Q} = 300 * DAYS$$

$$\frac{\partial v}{\partial Q} = 1.5333(150{,}000 * DAYS * Q^{-1})^{0.5333}(-150{,}000 * DAYS * Q^{-2})$$

Now we can calculate $\frac{d}{dt}\frac{\partial ELF}{\partial Q}$. Since ELF is a function of Q and X (or S), each of which is a function of time, the derivative will be as follows.

$$\frac{d}{dt}\frac{\partial ELF}{\partial Q} =$$

$$\underbrace{\frac{d}{dt}\{[(vu^{v-1})(\frac{du}{dQ}) + (\ln u)(u^{v})(\frac{dv}{dQ})]\}}_{\text{Unknown \#1}}\underbrace{[\frac{\partial f(Q)}{\partial Q}]}_{\text{known}} + \underbrace{\frac{d}{dt}\{[\frac{\partial f(Q)}{\partial Q}]\}}_{\text{unknown \#2}}\underbrace{[(vu^{v-1})(\frac{du}{dQ}) + (\ln u)(u^{v})(\frac{dv}{dQ})]}_{\text{known}}$$

First, we derive two equations that will be needed later in the following derivations.

$$u = g(f(Q)) = (1 - 300*DAYS*f(Q(t))) = (1 - 300 * DAYS * c_7Q(t)^{c8-1}X(t)^{c9})$$

$$\frac{du}{dt} = -300 * DAYS * c_7(c_8 - 1)Q(t)^{c_8-2}X(t)^{c_9}\frac{dQ(t)}{dt} - 300 * DAYS * c_7Q(t)^{c_8-1}c_9X(t)^{c_9-1}\frac{dX(t)}{dt}$$

$$\frac{du}{dt} = [-300 * DAYS * c_7][(c_8 - 1)Q(t)^{c_8-2}X(t)^{c_9}\frac{dQ(t)}{dt} + c_9Q(t)^{c_8-1}X(t)^{c_9-1}\frac{dX(t)}{dt}$$

$$v = (150{,}000*Days*Q^{-1})^{1.5333}$$

$$\frac{dv}{dt} = 1.5333(150{,}000 * DAYS * Q(t)^{-1})^{0.5333}(-150{,}000 * DAYS * Q(t)^{-2})\frac{dQ(t)}{dt}$$

Taking the unknown portions of the $\frac{d}{dt}\frac{\partial ELF}{\partial Q}$ equation in turn, we have the following:

Unknown #1:

$$\frac{d}{dt}[(vu^{v-1})(\frac{du}{dQ}) + (\ln u)(u^{v})(\frac{dv}{dQ})] = \underbrace{\frac{d}{dt}[(vu^{v-1})(\frac{du}{dQ})]}_{\text{Unknown A}} + \underbrace{\frac{d}{dt}[(\ln u)(u^{v})(\frac{dv}{dQ})]}_{\text{Unknown B}}$$

Unknown A

$$\frac{d}{dt}[(vu^{v-1})(\frac{du}{dQ})] = \underbrace{(\frac{d}{dt}(vu^{v-1}))}_{\text{Unknown a}}\underbrace{(\frac{du}{dQ})}_{\text{known}} + \underbrace{(\frac{d}{dt}(\frac{du}{dQ}))}_{\text{unknown b}}\underbrace{(vu^{v-1})}_{\text{known}}$$

Unknown a

$$\frac{d}{dt}(vu^{v-1}) = (v)(v-1)(u^{v-2})(\frac{du}{dt}) + (\ln(u))(u^{v-1})(\frac{dv}{dt})$$

Unknown b

$$\frac{du}{dQ} = 300 * DAYS \qquad => \qquad \frac{d}{dt}\frac{du}{dQ} = 0$$

<u>Unknown B</u>

$$\frac{d}{dt}[(\ln u)(u^v)(\frac{dv}{dQ})] = (\underbrace{\frac{d}{dt}(\ln u)}_{\text{Unknown c}})\underbrace{(u^v)}_{\text{known}}(\underbrace{\frac{dv}{dQ}}_{\text{known}}) + \underbrace{(\ln u)}_{\text{known}}(\underbrace{\frac{d}{dt}(u^v)}_{\text{unknwn d}})(\underbrace{\frac{dv}{dQ}}_{\text{known}}) + \underbrace{(\ln u)}_{\text{known}}\underbrace{(u^v)}_{\text{known}}(\underbrace{\frac{d}{dt}\frac{dv}{dQ}}_{\text{unknwn e}})$$

Unknown c

$$\frac{d}{dt}(\ln u) = \frac{1}{u}\frac{du}{dt}$$

Unknown d

$$\frac{d}{dt}(u^v) = vu^{v-1}\frac{du}{dt} + (\ln u)(u^v)(\frac{dv}{dt})$$

Unknown e

$$\frac{dv}{dQ} = \underbrace{1.533(150{,}000 * DAYS * Q(t)^{-1})^{0.533}}_{\text{j(Q(t)}}\underbrace{(-150{,}000 * DAYS * Q(t)^{-2})}_{\text{k(Q(t)}}$$

$$\frac{d}{dt}\frac{dv}{dQ}=1.533[0.533(150{,}000*DAYS*Q(t)^{-1})^{-0.466}(-150{,}000*DAYS*Q(t)^{-2})(\frac{dQ(t)}{dt})(-150{,}000*DAYS*Q(t)^{-2})$$
$$+(300{,}000*DAYS*Q(t)^{-3})(\frac{dQ(t)}{dt})(150{,}000*DAYS*Q(t)^{-1})^{0.533}]$$

Unknown C

$$\frac{df(Q)}{dQ}=\mathrm{c}_7(\mathrm{c}_8-1)\mathrm{Q(t)}^{\mathrm{c}_8-2}\mathrm{X(t)}^{\mathrm{c}_9}$$

$$\frac{d}{dt}\frac{df(Q)}{dQ}=\mathrm{c}_7(\mathrm{c}_8-1)[(\mathrm{c}_8-2)(\mathrm{Q(t)}^{\mathrm{c}_8-3})(\mathrm{X(t)}^{\mathrm{c}_9})(\frac{dQ(t)}{dt})+(c_9)(\mathrm{Q(t)}^{\mathrm{c}_8-2})(\mathrm{X(t)}^{\mathrm{c}_9-1})(\frac{dX(t)}{dt})$$

Unknown #2:

This unknown is the same as unknown 1-C, so we have the following,

$$\frac{d}{dt}\frac{df(Q)}{dQ}=\mathrm{c}_7(\mathrm{c}_8-1)[(\mathrm{c}_8-2)(\mathrm{Q(t)}^{\mathrm{c}_8-3})(\mathrm{X(t)}^{\mathrm{c}_9})(\frac{dQ(t)}{dt})+(c_9)(\mathrm{Q(t)}^{\mathrm{c}_8-2})(\mathrm{X(t)}^{\mathrm{c}_9-1})(\frac{dX(t)}{dt})$$

And now we have $\frac{d}{dt}\frac{\partial ELF}{\partial Q}$ entirely in terms of known functions

$$\frac{d}{dt}\frac{\partial ELF}{\partial Q}=\{\underbrace{\frac{d}{dt}[(vu^{v-1})(\frac{du}{dQ})]}_{\text{Known (1, A.a, A.b)}}+\underbrace{\frac{d}{dt}[(\ln u)(u^{v})(\frac{dv}{dQ})]}_{\text{Known (1, B.c, B.d, B.e)}}\}\underbrace{[\frac{\partial f(Q)}{\partial Q}]}_{\text{known}}+\{\underbrace{\frac{d}{dt}[\frac{\partial f(Q)}{\partial Q}]}_{\text{Known (2)}}\}\underbrace{[(vu^{v-1})(\frac{du}{dQ})+(\ln u)(u^{v})(\frac{dv}{dQ})]}_{\text{known}}$$

Derivation of Step 1 of Boundary Value Problem for Model Specifications including Royalty and Severance Tax

The term $\frac{d}{dt}\frac{\partial ELF}{\partial Q}$ contains the term $\frac{dQ}{dt}$. Thus, we need to expand $\frac{d}{dt}\frac{\partial ELF}{\partial Q}$ when doing the algebra to isolate $\frac{dQ}{dt}$ in step 1 of the Boundary Value Problem. In this appendix, we do this expansion for the variable discount rate case modeled with reserves remaining (S). Fortunately, all lettered terms other than "E" will remain the same for all other model specifications (i.e., constant discount rate and modeling with cumulative stock extracted (X) or no stock effects (no S or X). The term "E" unique to each model specification is defined in the main text of this report.

The lettered expressions u and v, and function f(Q) are defined in the appendix showing the derivation of $\frac{d}{dt}\frac{\partial ELF}{\partial Q}$.

Expanding the $\frac{d}{dt}\frac{\partial ELF}{\partial Q}$ term in the equation we last had in the main text before coming to this appendix, we have:

$$(1\text{-}LR_i\text{-}ST_iELF(Q_i(t)))\frac{d}{dt}P(t) - ST_iP(t)\frac{\partial ELF}{\partial Q}\frac{dQ}{dt} - ST_i\frac{dQ}{dt}P(t)\frac{\partial ELF}{\partial Q} - ST_iQ_{it}\frac{d}{dt}P(t)\frac{\partial ELF}{\partial Q} - \frac{\partial^2 C}{\partial Q^2}\frac{dQ}{dt} - \frac{\partial^2 C}{\partial S\partial Q}\frac{dS}{dt}$$

$$- ST_iQ_{it}P(t)\Big[\{[(v)(v-1)(u^{v-2})\frac{du}{dt} + (\ln u)(u^{v-1})\frac{dv}{dt}][\frac{du}{dQ}] + (0)(vu^{v-1}) + [(\frac{1}{u})(\frac{du}{dt})(u^{v})(\frac{dv}{dQ}) + (\ln u)(vu^{v-1}\frac{du}{dt} + (\ln u)(u^{v})\frac{dv}{dt})(\frac{dv}{dQ})$$

$$+ (\ln u)(u^{v})(\frac{d}{dt}\frac{dv}{dQ})]\}\{\frac{df(Q)}{dQ}\} + \{\frac{d}{dt}\frac{df(Q)}{dQ}\}\{(vu^{v-1})(\frac{du}{dQ}) + (\ln u)(u^{v})(\frac{dv}{dQ})\}\Big] = \frac{dC}{dS} + (P(t) - \frac{dC}{dQ})(\frac{dr(t)}{dt}t + r(t))$$

Isolating terms with $\frac{dQ}{dt}$ on the left side yields:

$$ST_iP(t)\frac{\partial ELF}{\partial Q}\frac{dQ}{dt} + ST_i\frac{dQ}{dt}P(t)\frac{\partial ELF}{\partial Q} + \frac{\partial^2 C}{\partial Q^2}\frac{dQ}{dt} + ST_iQ_{it}P(t)(v)(v-1)(u^{v-2})\frac{du}{dt}\frac{du}{dQ}\frac{df(Q)}{dQ} + ST_iQ_{it}P(t)(\ln u)(u^{v-1})\frac{dv}{dt}\frac{du}{dQ}\frac{df(Q)}{dQ}$$

$$+ ST_iQ_{it}P(t)(\frac{1}{u})(\frac{du}{dt})(u^{v})(\frac{dv}{dQ})\frac{df(Q)}{dQ} + ST_iQ_{it}P(t)(\ln u)(vu^{v-1})\frac{du}{dt}\frac{dv}{dQ}\frac{df(Q)}{dQ} + ST_iQ_{it}P(t)(\ln u)(\ln u)(u^{v})\frac{dv}{dt}\frac{dv}{dQ}\frac{df(Q)}{dQ}$$

$$+ \mathrm{ST_iQ_{it}P(t)(ln\ u)(u^v)}\frac{d}{dt}\frac{dv}{dQ}\frac{df(Q)}{dQ} + \mathrm{ST_iQ_{it}P(t)}\frac{d}{dt}\frac{df(Q)}{dQ}(\mathrm{vu^{v-1}})(\frac{du}{dQ}) + \mathrm{ST_iQ_{it}P(t)}\frac{d}{dt}\frac{df(Q)}{dQ}(\mathrm{ln\ u})(\mathrm{u^v})(\frac{dv}{dQ})$$

$$= (1\text{-}\mathrm{LR_i}\text{-}\mathrm{ST_iELF(Q_i(t))})\frac{d}{dt}\mathrm{P(t)} - \mathrm{ST_iQ_{it}}\frac{d}{dt}\mathrm{P(t)}\frac{\partial ELF}{\partial Q} - \frac{\partial^2 C}{\partial S\partial Q}\frac{dS}{dt} - \frac{dC}{dS} - (\mathrm{P(t)} - \frac{dC}{dQ})(\frac{dr(t)}{dt}\mathrm{t} + \mathrm{r(t)})$$

Collecting terms yields:

$$\frac{dQ}{dt}[2\mathrm{ST_iP(t)}\frac{\partial ELF}{\partial Q} + \frac{\partial^2 C}{\partial Q^2}]$$

$$+ \frac{du}{dt}[\mathrm{ST_iQ_{it}P(t)}\frac{df(Q)}{dQ}][(\mathrm{v})(\mathrm{v}\text{-}1)(\mathrm{u^{v-2}})\frac{du}{dQ} + (\frac{1}{u})(\mathrm{u^v})(\frac{dv}{dQ}) + (\mathrm{ln\ u})(\mathrm{u^{v-1}})\frac{dv}{dQ}]$$

Brackets [] are Expression A

$$+ \frac{dv}{dt}[\mathrm{ST_iQ_{it}P(t)}\frac{df(Q)}{dQ}][(\mathrm{ln\ u})(\mathrm{u^{v-1}})\frac{du}{dQ} + (\mathrm{ln\ u})(\mathrm{ln\ u})(\mathrm{u^v})\frac{dv}{dQ}]$$

Brackets [] are Expression B

$$+ \frac{d}{dt}\frac{dv}{dQ}[\mathrm{ST_iQ_{it}P(t)}\frac{df(Q)}{dQ}][(\mathrm{ln\ u})(\mathrm{u^v})]$$

Brackets [] are Expression C

$$+ \frac{d}{dt}\frac{df(Q)}{dQ}[\mathrm{ST_iQ_{it}P(t)}][(\mathrm{vu^{v-1}})(\frac{du}{dQ}) + (\mathrm{ln\ u})(\mathrm{u^v})\frac{dv}{dQ}]$$

Brackets [] are Expression D

$$= (1\text{-}\mathrm{LR_i}\text{-}\mathrm{ST_iELF(Q_i(t))})\frac{d}{dt}\mathrm{P(t)} - \mathrm{ST_iQ_{it}}\frac{d}{dt}\mathrm{P(t)}\frac{\partial ELF}{\partial Q} - \frac{\partial^2 C}{\partial S\partial Q}\frac{dS}{dt} - \frac{dC}{dS} - (\mathrm{P(t)} - \frac{dC}{dQ})(\frac{dr(t)}{dt}\mathrm{t} + \mathrm{r(t)})$$

Exp. E

Known equations from the derivation of $\frac{d}{dt}\frac{\partial ELF}{\partial Q}$ include the following (see appendix G):

$$\frac{du}{dt} = -300 * DAYS * c_7(c_8 - 1)Q(t)^{c_8 - 2}X(t)^{c_9}\frac{dQ(t)}{dt} - 300 * DAYS * c_7 Q(t)^{c_8 - 1}c_9 X(t)^{c_9 - 1}\frac{dX(t)}{dt}$$

$$\frac{dv}{dt} = 1.5333(150{,}000 * DAYS * Q(t)^{-1})^{0.5333}(-150{,}000 * DAYS * Q(t)^{-2})\frac{dQ(t)}{dt}$$

All except $\frac{dQ(t)}{dt}$ is Expression G

$$\frac{d}{dt}\frac{dv}{dQ} = 1.533[0.533(150{,}000 * DAYS * Q(t)^{-1})^{-0.466}(-150{,}000 * DAYS * Q(t)^{-2})(\frac{dQ(t)}{dt})(-150{,}000 * DAYS * Q(t)^{-2})$$

$$+(300{,}000 * DAYS * Q(t)^{-3})(\frac{dQ(t)}{dt})(150{,}000 * DAYS * Q(t)^{-1})^{0.533}]$$

$$= \frac{dQ}{dt}[0.8177(150{,}000 * DAYS * Q^{-1})^{-0.46}(150{,}000 * DAYS * Q^{-2})^{2} + 1.5333(300{,}000 * DAYS * Q^{-3})(150{,}000 * DAYS * Q^{-1})^{0.5333}$$

All except $\frac{dQ(t)}{dt}$ is Expression F

$$\frac{d}{dt}\frac{df(Q)}{dQ} = \mathrm{c}_7(\mathrm{c}_8 - 1)[(\mathrm{c}_8 - 2)(\mathrm{Q(t)}^{\mathrm{c}_8 - 3})(\mathrm{X(t)}^{\mathrm{c}_9})(\frac{dQ(t)}{dt}) + (c_9)(\mathrm{Q(t)}^{\mathrm{c}_8 - 2})(\mathrm{X(t)}^{\mathrm{c}_9 - 1})(\frac{dX(t)}{dt})$$

Then we can expand the differential terms in front of expressions A – D as follows:

$$\frac{dQ}{dt}\underbrace{[2\mathrm{ST_iP(t)}\frac{\partial ELF}{\partial Q} + \frac{\partial^2 C}{\partial Q^2}]}_{\textbf{Expression H}} + \frac{dQ}{dt}\underbrace{[-300*\mathrm{Days}*\mathrm{c}_7(\mathrm{c}_8-1)\mathrm{Q}^{\mathrm{c}8-2}\mathrm{X}^{\mathrm{c}9}]}_{\textbf{Expression I}}[\mathbf{A}] + \frac{dQ}{dt}[\mathbf{G}][\mathbf{B}] + \frac{dQ}{dt}[\mathbf{F}][\mathbf{C}] + \frac{dQ}{dt}\underbrace{[\mathrm{c7(c8-1)(c8-2)(Q^{c8-3})X^{c9}}]}_{\textbf{Expression J}}[\mathbf{D}]$$

$$= \underbrace{[300*\mathrm{Days}*\mathrm{c}_7*\mathrm{c}_9*\mathrm{Q}^{\mathrm{c}8-1}\mathrm{X}^{\mathrm{c}9-1}\frac{dX}{dt}]}_{\textbf{Expression K}}[\mathbf{A}] - \underbrace{[\mathrm{c}_7(\mathrm{c}_8-1)\mathrm{c}_9\mathrm{Q}^{\mathrm{c}8-2}\mathrm{X}^{\mathrm{c}9-1}\frac{dX}{dt}]}_{\textbf{Expression L}}[\mathbf{D}] + \mathbf{E}$$

Then, $\frac{dQ}{dt}$ [H + IA + GB + FC + JD] = KA – LD + E

Finally, $\frac{dQ}{dt}$ can be written in terms of the expressions defined above as follows:

$$\frac{dQ}{dt} \; \frac{KA - LD + E}{H + IA + GB + FC + JD}$$